CLASSICAL DYNAMICS

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Small amplitude oscillations

Part-II

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Small oscillation with generalised coordinates

Let us consider a general problem of a conservative system with *n* degrees of freedom and $\{q_i\}$ as the generalised coordinates, where i= *1,2, ... ,n*. Since the system is conservative, the forces acting on the system are derivable from a potential energy function V $(q_1, q_{2,} \ . \ . \ . \ , q_N)$. Without loss of generality, let us choose the equilibrium position at the origin $(q_1 = q_2 =. \ . \ . = q_N = 0)$. If the system is disturbed to a configuration $\{q_i\}$, we can write,

$$V(q_1, q_2, \ldots) = V(0, 0, \ldots) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j + \text{ higher order terms} \quad (1)$$

Where the subscript '0' refers to the equilibrium value.

The first term on the right hand side represents the equilibrium potential energy and hence may be considered zero by shifting the arbitrary zero of the potential.

The second term vanishes as $\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$ in equilibrium.

The only physically significant non-vanishing lowest order term is the third term. The smallness of oscillations makes the higher order terms totally insignificant.

Small oscillation with generalised coordinates

Let us write $$V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \quad (2)$$

So that $$V = \sum_{i,j} \frac{1}{2} V_{ij} q_i q_j \quad (3)$$

as a first approximation.

Now as V_{ij} is symmetric $V_{ij} = V_{ji}$.

Now, the kinetic energy of a scleronomic (when the generalised coordinates of the system do not involve time explicitly) system is, in general, a quadratic in generalised velocities, and can be written as

$$T = \frac{1}{2} \sum_{i,j} T_{ij} \dot{q}_i \dot{q}_j \quad (4)$$

where the coefficients T_{ij} are, in general, functions of generalised coordinates. One can expand T_{ij} in a Taylor series about the equilibrium position

$$T_{ij}(q_1, q_2, \ldots) = T_{ij}(0, 0, \ldots) + \sum_k \left(\frac{\partial T_{ij}}{\partial q_k}\right)_0 q_k + \ldots \quad (5)$$

Small oscillation with generalised coordinates

It turns out that the quantities $\left(\frac{\partial T_{ij}}{\partial q_k}\right)_0$ and the higher order derivatives are negligibly small so that the coefficients T_{ij}' s can be essentially treated as constants having the same values as they would have in the equilibrium position. Thus around the equilibrium position, the Lagrangian of the system is given by

$$L = T - V = \frac{1}{2}\sum_{i,j}\left(T_{ij}\dot{q}_i\dot{q}_j - V_{ij}q_iq_j\right) \qquad (6)$$

The Lagrange's equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

$$\frac{d}{dt}\sum_{i,j}\frac{1}{2}\left[T_{ij}\delta_{ik}\dot{q}_j + T_{ij}\dot{q}_i\delta_{kj}\right] + \frac{1}{2}\sum_{i,j}V_{ij}\left(\delta_{ik}q_j + q_i\delta_{kj}\right) = 0$$

$$\frac{1}{2}\left[\sum_j T_{kj}\ddot{q}_j + \sum_i T_{ik}\ddot{q}_i\right] + \frac{1}{2}\left[\sum_j V_{ik}q_j + \sum_i V_{ik}q_i\right] = 0 \qquad (7)$$

Changing the dummy summation index j to i in the first and the third terms of the above and using the symmetry of V_{ij} and of T_{ij}, we get

$$\sum_i T_{ij}\ddot{q}_i + \sum_i V_{ik}q_i = 0 \quad \text{for each } k. \qquad (8)$$

Small oscillation with generalised coordinates

We seek a solution to this equation of the form $q_i = A_i e^{i\omega t}$

which gives $$\sum_i \left(V_{ik} - \omega^2 T_{ik}\right) A_i = 0 \tag{9}$$

The equation is a homogeneous equation in A_is and the condition for existence of the solution is

$$\det(V_{ik} - \omega^2 T_{ik}) = 0 \tag{10}$$

which is a single algebraic equation of n-th degree in ω^2. This equation has n roots some of which are real and some complex (some of the roots may be degenerate, i.e., may be same for two or more values of *k*). We are only interested in real roots of the above equation. ω_k's determined from this equation are known as characteristic frequencies or eigenfrequencies.

From physical arguments it is clear that for real physical situations, the roots are real and positive. This is because the existence of an imaginary part in ω would mean time dependence of q_k and $\dot{q}_k$ such that the total energy would not be conserved in time and such solutions are unacceptable.

Small oscillation with generalised coordinates

Let us rewrite (9) (using symmetry properties of V and T) as

$$\sum_i (V_{ki} - \lambda T_{ki}) A_i = 0 \qquad (11)$$

Let us define a column vector

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_N \end{pmatrix}$$

The matrices V and T are given by

$$V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1N} \\ V_{21} & V_{22} & \dots & V_{2N} \\ \dots & \dots & \dots & \dots \\ V_{N1} & V_{N2} & \dots & V_{NN} \end{pmatrix} \qquad T = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{21} & T_{22} & \dots & T_{2N} \\ \dots & \dots & \dots & \dots \\ T_{N1} & T_{N2} & \dots & T_{NN} \end{pmatrix}$$

Using these, the equation (9) can be expressed as a matrix equation

$$VA = \omega^2 TA \qquad (12)$$

Note that this equation is not in the form of an eigenvalue equation as VA is not equal to a constant times A but a constant times TA. (If T is invertible, one can get an eigenvalue equation

$$T^{-1} VA = \omega^2 IA. \qquad (13)$$

Small oscillation with generalised coordinates

Since we have *N* homogeneous equations, we have *N* modes, i.e. *N* solutions for ω^2.. Let us denote the *k*-th mode frequency by $\omega_k^2 = \lambda_k$. Let the vector *A* corresponding to this mode be written as

$$A_k = \begin{pmatrix} A_{k1} \\ A_{k2} \\ \dots \\ A_{kN} \end{pmatrix}$$

We then have
$$VA_k = \lambda_k T A_k \tag{14}$$

Taking conjugate of this equation and changing the index *k* to *i*, we get

$$\tilde{A}_i V = \lambda_i \tilde{A}_i T \tag{15}$$

where we have used $\tilde{A}$ to denote the transpose of the matrix A . From (14) we get by multiplying with $\tilde{A}_k$

$$\lambda_k = \frac{\tilde{A}_k V A_k}{\tilde{A}_k T A_k} \tag{16}$$

From (15) and (16) it follows that

$$\tilde{A}_i V A_k = \lambda_k \tilde{A}_i T A_k$$
$$\tilde{A}_i V A_k = \lambda_i \tilde{A}_i T A_k$$

Small oscillation with generalised coordinates

so that

$$(\lambda_k - \lambda_i)\tilde{A}_i T A_k = 0 \quad (17)$$

Thus, if the eigenvalues are non-degenerate, i.e.,if $\lambda_i \neq \lambda_k,$ we get the orthogonality condition

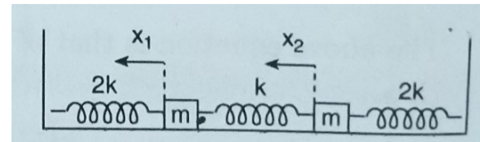
$$\tilde{A}_i T A_k = 0 \quad (18)$$

Note that this is different from the orthogonality condition on eigen vectors for a regular eigenvalue equation. Since (12) does not uniquely determine *A*, we define normalization condition as $\tilde{A}_i T A_i = 1$

Examples: Small oscillation

Problem: Two blocks of equal masses (m) are joined with a spring as shown in figure. They execute small oscillations on a frictionless surface. Find the normal frequencies of the oscillating system.

Solution: Let x_1 and x_2 be the extensions of springs of force constant 2k. Then the kinetic energy (T) and the potential energy (V) may be written as



$$T = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

$$\begin{aligned} V &= \tfrac{1}{2}(2k)x_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}(2k)x_2^2 \\ &= \tfrac{3}{2}kx_1^2 + \tfrac{3}{2}kx_2^2 - kx_1x_2 \end{aligned}$$

$$\therefore \quad \boldsymbol{T} = (T_{ij}) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\boldsymbol{V} = (V_{ij}) = \begin{pmatrix} 3k & -k \\ -k & 3k \end{pmatrix}$$

$$\therefore \quad \left|\boldsymbol{V} - \omega^2\boldsymbol{T}\right| = \begin{vmatrix} 3k - m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0$$

$$\text{or,} \quad (3k - m\omega^2)^2 - k^2 = 0$$

$$\text{or,} \quad 3k - m\omega^2 = \pm k$$

$$\omega^2 = \frac{3k \mp k}{m}$$

$$\therefore \quad \omega_1 = \sqrt{\frac{2k}{m}}; \quad \omega_2 = \sqrt{\frac{4k}{m}}$$

These are the normal frequencies of oscillation of the system

Examples: Small oscillation

Problem: Two masses, each equal to m, are connected by massless springs of spring constant k such that they can freely slide on a smooth horizontal surface. The ends of the springs are fixed to vertical walls. Determine (i) the normal frequencies (ii) the normal modes of vibrations and (iii) the normal coordinates.

Solution:

(i) Let η_1 and η_2 be the displacements of the masses from the equilibrium position.

The kinetic energy of the system is

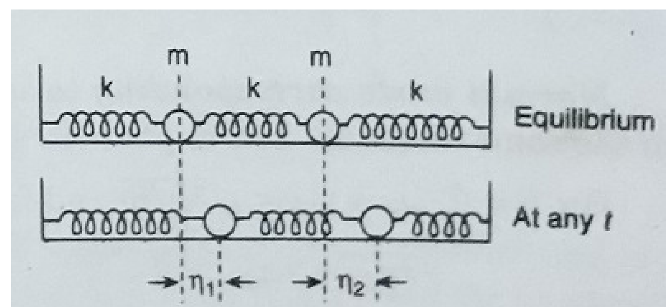
$$T = \tfrac{1}{2}m\dot{\eta}_1^2 + \tfrac{1}{2}m\dot{\eta}_2^2 = \tfrac{1}{2}m(\dot{\eta}_1 \ \ \dot{\eta}_2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\eta}_1\\ \dot{\eta}_2\end{pmatrix}$$

$$\therefore \ \ T = m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}$$

The potential energy of the system is given by

$$\begin{aligned} V &= \tfrac{1}{2}k\eta_1^2 + \tfrac{1}{2}\eta_2^2 + \tfrac{1}{2}k(\eta_2 - \eta_1)^2 \\ &= \tfrac{1}{2}k(2\eta_1^2 + 2\eta_2^2 - 2\eta_1\eta_2) \\ &= \tfrac{1}{2}(\eta_1 \ \ \eta_2)\begin{pmatrix}2k & -k\\ -k & 2k\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \end{aligned}$$

$$\therefore \ \ V = \begin{pmatrix}2k & -k\\ -k & 2k\end{pmatrix}$$



Examples: Small oscillation

The characteristic equation is given by

$$|V - \omega^2 T| = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k. & 2k - m\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2k - m\omega^2)^2 - k^2 = 0$$

$$\Rightarrow \quad 2k - m\omega^2 = \pm k$$

$$\therefore \quad \omega_1 = \sqrt{k/m}; \quad \omega_2 = \sqrt{3k/m}$$

Which are the two normal frequencies.

(ii) We have, $(V - \omega^2 T)a = 0,$ or $\sum_j (V_{ij} - \omega_k^2 T_{ij}) a_{jk} = 0, \ i = 1, 2$

For normal modes, corresponding to ω_k, we have

$$\begin{vmatrix} 2k - m\omega_k^2 & -k \\ -k & 2k - m\omega_k^2 \end{vmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \end{pmatrix} = 0 \qquad (1)$$

Alongwith the normalization condition $\tilde{a} T a = 1$ or, $\sum_{ij} T_{ij} a_{il} a_{jk} = \delta_{lk}.$

In this case, $T_{ij} = 0$ for $i \neq j$ and $\delta_{lk} = 0$ for $l \neq k.$

The normalization condition becomes

$$\sum_i T_{ii} a_{ik}^2 = 1 \quad \Rightarrow \quad m(a_{1k}^2 + a_{2k}^2) = 1 \qquad (2)$$

Examples: Small oscillation

For k=1 $\omega_k = \omega_1 = \sqrt{k/m}$

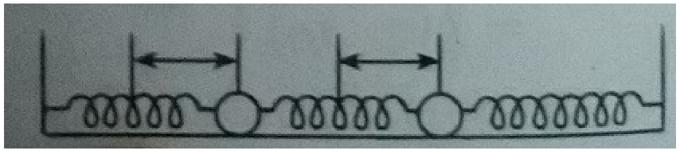
Thus, from (1)
$$\begin{aligned} ka_{11} - ka_{21} &= 0 \\ -ka_{11} + ka_{21} &= 0 \end{aligned} \quad \Longrightarrow \quad a_{11} = a_{21}$$

Thus the amplitudes are not uniquely specified for a particular frequency (ω_1), but their ratio is unique (here 1). The values of amplitudes are arbitrarily fixed using the normalization condition (2).

$$m(a_{11}^2 + a_{21}^2) = 1 \quad \Longrightarrow \quad a_{11} = a_{21} = 1/\sqrt{2m}$$

Eigenvector corresponding to ω_1 is
$$\boldsymbol{a}_1 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normal mode corresponding to ω_1 thus corresponds to the motion of the masses in the same direction (i,.e., in phase).



Examples: Small oscillation

For k=2, $\omega_k = \omega_2 = \sqrt{3k/m}$

From (1)

$$-ka_{12} - ka_{22} = 0$$
$$-ka_{12} - ka_{22} = 0$$

$\Rightarrow$ $a_{12} = -a_{22}$

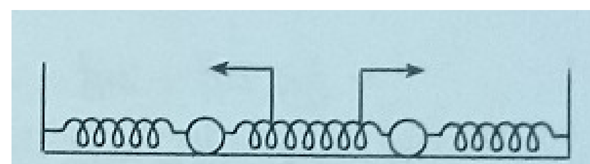
Here also the ratio of the amplitudes is fixed (= -1), while the absolute values are not.

From the normalization condition we get,

$m(a_{12}^2 + a_{22}^2) = 1$ $\Rightarrow$ $a_{12} = -a_{22} = 1/\sqrt{2m}$

Eigenvector corresponding to ω_2 is $a_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The normal mode corresponding to ω_2 thus corresponds to motion of the masses in opposite direction (out-of-phase).



Note that the eigenvectors a_1 and a_2 are orthogonal $\Rightarrow$ $a_1 \cdot a_2 = 0.$

Examples: Small oscillation

(iii) We have

$$a = 1/\sqrt{2m}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$a^{-1} = \frac{\text{Adj}\,(a)}{|a|} = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Thus

$$\xi = a^{-1}\eta$$

Hence,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Normal coordinates in terms of η_1 and η_2

$$\xi_1 = \sqrt{\frac{m}{2}}(\eta_1 + \eta_2)$$

$$\xi_2 = \sqrt{\frac{m}{2}}(\eta_1 - \eta_2)$$

Examples: Small oscillation

Problem: Two similar springs, each of spring constant k, hang vertically downward from a rigid support with two equal masses m attached to them as shown in figure. Find the normal mode of frequencies of small oscillations along the vertical direction only.

Solution:

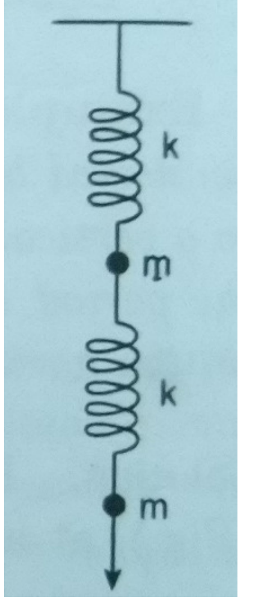
Let u_1 and u_2 be the displacements of the two equal masses from their equilibrium positions.

Hence, Kinetic energy $T = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2)$

and Potential energy
$$V = \frac{1}{2}k\left[u_1^2 + (u_2 - u_1)^2\right] = \frac{1}{2}k(2u_1^2 + u_2^2 - 2u_1u_2)$$

Thus,
$$2T = (\dot{u}_1\ \dot{u}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}$$

$$2V = (u_1\ u_2)\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



Examples: Small oscillation

Kinetic and Potential energy matrices are therefore

$$[T_{ij}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$ and $$[V_{ij}] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

The characteristic equation is

$$\left|V - \omega^2 T\right| = \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = 0$$

$$\text{or,}\quad (2k - \omega^2 m)(k - \omega^2 m) - k^2 = 0$$

$$\text{or,}\quad \omega^4 m^2 - 3\omega^2 mk + k^2 = 0$$

$$\text{or,}\quad \omega^2 = \frac{3mk \pm \sqrt{9m^2k^2 - 4m^2k^2}}{2m^2} = \frac{3}{2}\frac{k}{m} \pm \sqrt{5}\frac{k}{2m}$$

Hence $$\omega_1 = \sqrt{\frac{k}{2m}(3 + \sqrt{5})}$$

and $$\omega_2 = \sqrt{\frac{k}{2m}(3 - \sqrt{5})}$$

These are the normal mode frequencies.

Suggested readings:

1. *Classical Mechanics*, H. Goldstein, C.P. Poole, J.L. Safko, 3rd Edn. 2002, Pearson Education
2. *Mechanics*, L. D. Landau and E. M. Lifshitz, 1976, Pergamon
3. Classical Mechanics, J C Upadhyay, 2014, Himalaya Publishing House Pvt,. Ltd.

For practice, follow examples of any other text books on Classical Mechanics