Economics Honours (Semester II)

Mathematical Economics - I

Economic Examples in Difference and Differential Equations

Section I: Use of Difference Equations in Economics

Lagged Income Determination Model

Assume that consumption is a function of the previous period's income, so that

$C_t = C_0 + cY_{t-1}$ and $Y_t = C_t + I_t$ where $I_t = I_0$. Thus $Y_t = C_0 + cY_{t-1} + I_0$

Rearranging the terms

$$Y_t = cY_{t-1} + C_0 + I_0 \qquad (1)$$

This can be written in the form of the difference equation $y_t = by_{t-1} + a$, where b and a are constants and the general solution is

$$y_t = [y_0 - a/(1-b)]\, b^t + [a/(1-b)] \qquad \text{when} \quad b \neq 1 \qquad (2)$$

$$y_t = y_0 + at \qquad \text{when} \quad b = 1 \qquad (3)$$

Thus (1) can be rewritten as $Y_t = bY_{t-1} + a$ [where c=b and $a = C_0 + I_0$] (4)

In the equation (4) b is the marginal propensity to consume (c) which cannot be equal to 1. Thus assuming $Y_t = Y_0$ at t = 0, the solution is given as

$$Y_t = [Y_0 - (C_0 + I_0)/(1-c)]\, c^t + (C_0 + I_0)/(1-c)$$

The stability of the time path thus depends on c. Since $0 < MPC < 1$, $|c| < 1$ and the time path will converge. Since $c > 0$, there will be no oscillations. The equilibrium is stable, and as $t \to \infty$, $Y_t \to (C_0 + I_0)/(1-c)$ which is the intertemporal equilibrium level of income.

Example 1: Given $Y_t = C_t + I_t$, $C_t = 200 + 0.9Y_{t-1}$ and $I_t = 100$ and $Y_0 = 4500$. Solve for Y_t

Here the consumption equation is given in the form $C_t = C_0 + cY_{t-1}$ where $C_0 = 200$ and c =0.9

The income equation is given as $Y_t = C_t + I_t$

$$\text{i.e. } Y_t = C_0 + cY_{t-1} + I_0 \quad [\text{where } I_t = I_0]$$

$$\text{i.e. } Y_t = 200 + 0.9Y_{t-1} + 100 = 0.9Y_{t-1} + 300$$

which is in the form of the difference equation $Y_t = bY_{t-1} + a$ where a = 300 and b= c =MPC =0.9.

The solution of the difference equation is as follows:

$$Y_t = [Y_0 - (a)/(1- b)]\, b^t + (a)/(1-b)$$

Substituting the values

$$Y_t = [4500 - (300)/(1- 0.9)]\, 0.9^t + (300)/(1-0.9)$$

$$\text{i.e. } Y_t = 1500(0.9)^t + 3000$$

With $|0.9| < 1$ and the time path will converge. Since $0.9 > 0$, there will be no oscillations. The equilibrium is stable, and as $t \to \infty$, the first term on the right hand side goes to zero and $Y_t \to 3000$ which is the intertemporal equilibrium level of income.

The Cobweb Model

It has been found that in certain products like agricultural commodities, current supply depends upon previous year's price level which can be studied with the help of the Cobweb Model.

If the demand function is $Q_{dt} = c + bP_t$

And supply function is $Q_{st} = g + hP_{t-1},$

then in equilibrium where demand is equal to supply

$$Q_{dt} = Q_{st}$$

$$\text{i.e. } c + bP_t = g + hP_{t-1}$$

$$\text{i.e. } bP_t = hP_{t-1} + g - c$$

$$\text{i.e. } P_t = h/b(P_{t-1}) + (g - c)/b \text{ -----------(5)}$$

which is in the form of the difference equation $y_t = by_{t-1} + a$, where b and a are constants.

In the above difference equation (5) $b = h/b$ and $a = (g - c)/b$ -----------(6)

Under normal demand supply conditions, we know that the demand curve is downward sloping, hence $b < 0$ and the supply curve is upward rising, hence $h > 0$ Thus $h/b \neq 1$.

The solution to the difference equation $y_t = by_{t-1} + a$ when $b \neq 1$ is

$$y_t = [y_0 - a/(1- b)]\, b^t + a/(1-b)$$

Thus the solution to the difference equation in (5) is

$$P_t = [P_0 - (g-c)/(b-h)](h/b)^t + (g-c)/(b-h) \text{ ----------(7)}$$

When the Cobweb model is in equilibrium, $P_t = P_{t-1}$. Suppose we write

$P_e = P_t = P_{t-1}$, then $P_e = (g-c)/(b-h)$ ----------------------(8).

Using (8) in (7),

$$P_t = [P_0 - P_e](h/b)^t + P_e$$

With an ordinary negative demand function and positive supply function, $b < 0$ and $h > 0$, therefore $h/b < 0$ and the time path will oscillate.

If $|h| > |b|$, i.e. $|h/b| > 1$, the time path P_t will explode

If $|h| = |b|$, $h/b = -1$, the time path oscillates uniformly

If $|h| < |b|$, i.e. $|h/b| < 1$, the time path will converges and P_t approaches P_e.

When Q =f(P) in the supply-demand analysis, the supply curve must be flatter than the demand curve for stability. But if P = f(Q), then the demand curve must be flatter or more elastic than the supply curve if the model is to be stable.

Example 2: Given $Q_{dt} = 86 - 0.8P_t$ and $Q_{st} = -10 + 0.2 P_{t-1}$, the market price P_t for nay time period and equilibrium price P_e can be found as follows. Equating demand and supply

$$86 - 0.8 P_t = -10 + 0.2 P_{t-1}$$

$$\text{i.e. } - 0.8 P_t = + 0.2 P_{t-1} - 96$$

$$\text{i.e. } P_t = - 0.25 P_{t-1} + 120$$

Using (7) and (8), the solution to the difference equation is

$$P_t = (P_0 – 96)(- 0.25)^t + 96$$

And the equilibrium price P_e =96

Corresponding to the difference equation ($y_t= by_{t-1} + a$) as b = -0.25 which is negative and less than one, the time path oscillates and converges. The equilibrium is stable and as $t\to\infty$, P_t will converge to P_e = 96.

Section II: Use of Differential Equations in Economics

Example 3: Find the demand function Q =f (P) if point elasticity ε is -1 for all P > 0.

$$\varepsilon = dQ/dP \times P/Q = -1$$

$$\text{i.e. } dQ/dP = - Q/P$$

Separating the variables, we get

$$dQ/Q = - dP/P$$

Integrating both sides

$$\text{i.e. } \int dQ/Q = - \int dP/P$$

$$\text{i.e. } \log Q = - \log P + \log C$$

i.e. QP = C

Example 5: Derive the formula $P = P(0)e^{it}$ for the the total value of an initial sum of money P(0) set out for t years at interest rate i, when i is compounded continuously.

If i is compounded continuously,

$$dP/dt = iP$$

Separating the variables,

$$dP/P = idt$$

Integrating, $\log P - i.t = c$

Setting the LHS as an exponent of e,

$$e^{\log P - it} = c$$

$$Pe^{-it} = c$$

$$\text{i.e. } P = c\, e^{it}$$

At t =0, P = P(0). Thus $P(0) = c\, e^{0}$, c = P(0), and $P = P(0)e^{it}$.

*N.B. Explanations of all the theories and examples have been taken from the reference mentioned below. The examples given are not exhaustive.

References

Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.