Economics Honours (Semester II)

Mathematical Economics - I

Difference Equations

Box 1: *Definition of Difference Equations*

A difference equation expresses a relationship between a dependent variable and a lagged independent variable(s) which changes at discrete intervals of time, for example, $I_t = f(Y_{t-1})$, where I and Y are measured at the end of a year. The order of a difference equation is determined by the greatest number of periods lagged. A first-order difference equation expresses a time lag of one period; a second-order, two periods; etc. The change in y as t changes from t to t+1 is called the first difference of y. It is written as

$$\frac{\Delta y}{\Delta t} = \Delta y_t = y_{t+1} - y_t \text{------------(1)}$$

where Δ is an operator replacing d/dt that is used to measure continuous change in differential equations. The solution of a difference equation defines y for every value of t and does not contain a difference expression.

Example 1: Each of the following is a difference equation of the order indicated.

$I_t = a(Y_{t-1} - Y_{t-2})$	order 2
$Q_s = a + b\,P_{t-1}$	order 1
$y_{t+3} - 9y_{t+2} + 2y_{t+1} + 6y_t = 8$	order 3

Source: Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.

Notes:

Lagged means showing a delayed effect.

Δ is the symbol for difference operator

Δt is a finite quantity and represents difference interval

Box 2: *Determining the solution of a difference equation by the iterative method*

Example 2: Suppose we have a first order difference equation

$$\Delta y_t = 5y_t$$

As from (1), we know, $\Delta y_t = y_{t+1} - y_t$, we can write,

$$y_{t+1} - y_t = 5y_t$$

$$\text{i.e. } y_{t+1} = 5y_t + y_t$$

$$\text{i.e. } y_{t+1} = 6y_t \quad (2)$$

which can be written in the general form $y_{t+1} = by_t$ (3)

Given that the initial value of y is y_0, in the difference equation $y_{t+1} = by_t$ a solution is found as follows:

By successive substitution of t =0,1,2, 3, etc.in (3)

$$y_1 = by_0$$

$$y_2 = by_1 = b(by_0) = b^2y_0$$

$$y_3 = by_2 = b(b^2y_0) = b^3y_0$$

$$y_4 = by_3 = b(b^3y_0) = b^4y_0$$

Continuing in this manner for any period t,

$$y_t = b^ty_0$$

This method is called the iterative method. As y_0 is a constant, b plays a crucial role in determining values of y as t changes.

Source: Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.

Box 3: *General Formula for First-Order Linear Difference Equations*

Given a first-order difference equation which is linear (i.e., all variables are raised to the first power and there are no cross products)

$$y_t = by_{t-1} + a$$

where b and a are constants, the general formula for a *definite solution* is

$$y_t = [y_0 - a/(1-b)]\, b^t + \frac{a}{(1-b)} \quad \text{when} \quad b \neq 1 \qquad (4a)$$

$$y_t = y_0 + at \quad \text{when} \quad b = 1 \qquad (4b)$$

If no initial condition is given, an arbitrary constant A is used for $[y_0 - a/(1-b)]$ in (4a) and for y_0 in (4b). This is called a general solution**.**

Formula in (4a & 4b) can be expressed in the general form $y_t = Ab^t + c$ --------(5) where $A = [y_0 - a/(1-b)]$ and $c = a/(1-b)$.

Here Ab^t is called the *complementary function* and c is the *particular solution*. The *particular solution* expresses the intertemporal equilibrium level of y; the *complementary function* represents the deviations from that equilibrium.

Source: Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.

Note: The complementary function is also called the homogeneous solution which can be obtained by letting c =0.

Box 4: *Stability Conditions*

For a first-order linear difference equation of the form $y_t = by_{t-1} + a$

the definite solution is given by

$$y_t = Ab^t + c, \qquad (6)$$

where $A = [y_0 - a/(1-b)]$ and $c = a/(1-b)$.

The equation will be dynamically stable, therefore, only if the complementary function $Ab^t \rightarrow 0$ as $t \rightarrow \infty$. All depends on the base 'b' of the exponent b^t. On the right hand side of (6) assuming A =1 and c=0 for the moment, the exponential expression b^t will generate seven different time paths depending on the value of 'b'. The behaviour of the time path depending on the value of b can be stated in short as follows:

- If $|b| > 1$, the time path explodes
- If $|b| < 1$, the time path converges
- If $b > 0$, the time path is non-oscillating
- If $b < 0$, the time path oscillates
- If $b = -1$, the time path oscillates uniformly
- If $b = 1$, $b^t = 1$ for all values of t, the time path does not oscillate and is diagrammatically represented as a horizontal linear path.
- If $b = 0$, $b^t = 0$ for all values of t, the time path does not exist.

Source: Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.

*N.B. Explanations of all the theories and examples have been taken from the following reference:

Reference

Dowling, E. (2012). *Schaum's outline of introduction to mathematical economics* (3rd ed.). McGraw Hill Professional.