Economics Honours (Semester II)

Mathematical Economics -I

Differential Equations

Meaning of Differentials in Calculus

The simplest expression of a differential can be stated in this manner. If we have a function

$$y = f(x),$$

then, the derivative of y with respect to x is given as

$$dy/dx = f'(x)$$

On multiplying both sides of the equation by dx, we obtain

$$dy = f'(x)dx$$

This dy is known as the differential of y.

The question arises what is the interpretation? We had already presented the derivative dy/dx as a single symbol denoting the limit of $\Delta y/\Delta x$ as $\Delta x \rightarrow 0$. The derivative dy/dx can also be treated as a ratio of two separate symbols namely dy which is called the differential of y and dx which is called the differential of x. Thus given a function y = f(x), the differential of y, dy, measures the change in y resulting from a small change in x, written as dx. Thus

$$dy = f'(x).dx$$

i.e Change in y = (Rate at which y changes for a small change in x) . (a small change in x) .

For a given function y = f(x), differentials help us to approximate the change in y resulting from a small change in x. This is explained with the help of an example.

Example 1: With the help of differentials, approximate the change in the area of a square if the length of its side increases from 10 cm to 10.2 cm.

Suppose the area of a square is denoted by y and length of its side by x, then the area of the square is given by the function

$$y = x^2$$

$$\text{When } x = 10, y = 100$$

$$\text{If } x \text{ increases to } 10.2, y \text{ increases to } 104.04$$

$$\text{Thus } = (10.2 - 10.0) = 0.2 \text{ and } \Delta y = (104.04 - 100) = 4.04$$

Thus for a change in x, i.e. ($\Delta x = 0.2$), the corresponding change in y, i.e. $\Delta y = 4.04 = 4$ approximately.

The derivative of y with respect to x is given by

$$dy/dx = 2x$$

Then, the differential of y is given by

$$dy = 2x\, dx$$

Now, we know $\Delta x = dx = 0.2$, then when x=10, change in y i.e. the differential

$$dy = 2 \times 10 \times 0.2 = 4.0$$

This proves that the differential of y, dy = 4 is an approximation of the incremental value of y, i.e. $\Delta y = 4$.

Total and Partial Differentials

Suppose there is a function u = f(x,y). The *Total Differentials* give us a linear approximation of the small change in u = f(x,y) when there is a small change in both x and y. Let u denote paddy cultivation, x denote labour and y denote capital. Then total differential of u gives us an idea of the change in u or paddy cultivation when there is a small change in x, i.e., labour and y, i.e., capital. The partial derivative of u with respect to x is denoted by $\delta u/\delta x$; hence if x changes by Δx, then the change in u will be $(\delta u/\delta x).\Delta x$. Similarly when there is a change in y by Δy, the change in u will be $(\delta u/\delta y).\Delta y$. Thus as a linear approximation change in u due to a small change in x and y will be

$$\Delta u = (\delta u/\delta x).\Delta x + (\delta u/\delta y).\Delta y$$

Let us denote $\Delta u = dx$, $\Delta x = dx$ and $\Delta y = dy$. Then,

$$du = (\delta u/\delta x).dx + (\delta u/\delta y).dy$$

$$\text{i.e. } du = f_x\, dx + f_y\, dy$$

The du obtained is called the *total differential* of the function u = f(x,y) and is also denoted by df.

Example 2: $u = f(x,y) = 2x + 5y^2$.

The total differential is given by $du = f_x\, dx + f_y\, dy$

$$f_x = (\delta u/\delta x) = \delta(2x + 5y^2)/\delta x = 2$$

$$f_y = (\delta u/\delta y) = \delta(2x + 5y^2)/\delta y = 10y.$$

This when substituted in the total differential formula, gives,

$$du = 2dx + 10ydy$$

In the function u = f(x,y),

the *Partial Differential* of x gives the change in u resulting from a small change in either x, assuming y to be constant. Represented symbolically ,

$$du = f_x\, dx \quad \text{(assuming y constant)}$$

From Example 2,

$$du = 2dx \text{ (assuming y constant)}$$

and the *Partial Differential* y gives the change in u resulting from a small change in either y, assuming x to be constant. Represented symbolically,

$$du = f_y\, dy \text{ (assuming x constant)}$$

From Example 2,

$$du = 10ydy \text{ (assuming x constant)}$$

Meaning of Differential Equation

An equation which expresses implicitly or explicitly the relationship between a function and one or more of its derivative is called a differential equation. For example

$$dy/dx = 2x + 5$$

The equations which involve a single independent variable like above or in the form

$$dy/dx = f(x)$$

are called *ordinary differential equations.*

The *solution* to an ordinary differential equation is obtained by the *method of integration*. The solution is an ordinary equation without differentials or derivatives defined over an interval and satisfy the differential equation for all values of the independent variable. Given,

$$dy/dx = f(x)$$

$$\text{i.e. } dy = f(x)\, dx$$

Taking Integral on both sides,

$$\int dy = \int f(x)\, dx$$

$$\text{i.e. } y = \int f(x)\, dx$$

The *order* of the differential equation is the order of the highest derivative in the equation.

The *degree* of a differential equation is the highest power to which the derivative of highest order is raised.

Example 3: Solve the differential equation dy/dx = 2x +5

Let us rewrite the differential equation dy/dx = as

$$dy = (2x + 5)\, dx$$

Integrating both sides

$$\int dy = \int (2x + 5)\, dx$$

$$y = 2x^2/2 + 5x + C$$

$$\text{i.e. } y = x^2 + 5x + C$$

where C is the constant of integration.

Example 4: Find the order and degree of the following differential equation:

1. $dy/dt = 2t + 6$

2. $(dy/dt)^4 - 5t^5 = 0$

3. $d^2y/dt^2 + (dy/dt)^3 + t^2 = 0$

4. $(d^2y/dt^2)^7 + (d^3y/dt^3)^5 = 75y$

Solutions:

1. $dy/dt = 2t + 6$

The highest order of the derivative in the differential equation is of the first order. Hence the differential equation is of the first order.

The highest power to which the first order derivative (highest order in the given differential equation) is raised is the first degree. Hence the degree of a differential equation is first degree.

Solution: First order, First degree.

2. $(dy/dt)^4 - 5t^5 = 0$

The highest order of the derivative in the differential equation is of the first order. Hence the differential equation is of the first order.

The highest power to which the first order derivative (highest order in the given differential equation) is raised is fourth degree. Hence the degree of a differential equation is of the fourth degree.

Solution: First order, Fourth degree.

3. $d^2y/dt^2 + (dy/dt)^3 + t^2 = 0$

The highest order of the derivatives in the differential equation is of the second order. Hence the differential equation is of the second order.

The highest power to which the second order derivative (highest order in the given differential equation) is raised is first degree. Hence the degree of a differential equation is of the first degree.

Solution: Second order, First degree.

4. $(d^2y/dt^2)^7 + (d\,^3y/dt^3)^5 = 75y$

The highest order of the derivatives in the differential equation is of the third order. Hence the differential equation is of the third order.

The highest power to which the third order derivative (highest order in the given differential equation) is raised is fifth degree. Hence the degree of a differential equation is of the fifth degree.

Solution: Third order, Fifth degree.

First Order Homogenous Differential Equation

A first-order differential equation M(x,y) dx + N (x,y) dy =0 is said to be homogeneous if M(x,y) and N(x,y) are both homogeneous functions of the same degree.

[Note: A function f(x,y) is said to be homogeneous of degree n if the equation $f(tx,ty) = t^n f(x,y)$].

Example 5: Prove that the differential equation $(x – y)\, dx + (x^2 /y)\, dy =0$ is a first-order homogemeous differential equation of degree one.

The differential equation $(x – y)\, dx + (x^2 /y)\, dy =0$ is in the form

$$M(x,y)\, dx + N (x,y)\, dy =0.$$

$$\text{Suppose } M(x,y) = (x – y)$$

Then $M(tx, ty) = (tx – ty) = t(x – y)$ is a homogeneous function of degree one.

$$\text{Again suppose } N(x,y) = (x^2 /y)$$

Then $N(tx, ty) = (t^2 x^2 /ty) = t\,(x^2 /y)$ is homogeneous function of degree one.

As both $M(x, y) = (x – y)$ and $N(x, y) = (x^2 /y)$ are homogeneous functions of degree one, we conclude that the differential equation $(x – y)\, dx + (x^2 /y)\, dy =0$ is a first-order homogemeous differential equation of degree one.

Example 6: *Case of Exact Differential Equation*

Statement: Suppose there is a function z = f(x,y) and the first order differential equation is of the form M(x,y)dx + N(x,y)dy = 0, where M(x,y) and N(x,y) are partial derivatives of the function with respect to x and y respectively, then [δM/δy = δN/δx] is the necessary and sufficient condition for an expression to be an exact differential equation.

Problem: Show that the first order differential equation of the function $f(x,y) = x^2y$ is a homogenous and exact differential equation.

The first order differential equation of the function is

$$df = f_x dx + f_y dy = 2xydx + x^2dy$$

This means by the above statement $M(x,y) = f_x = 2xy$ and $N(x,y) = f_y = x^2$

Here as $M(tx,ty) = 2(tx)(ty) = t^2(2xy) = t^2 M(x,y)$; it is homogenous function of degree 2(two).

Again as $N(tx,ty) = (tx)^2 = t^2 x^2 = t^2 N(x,y)$; it is homogenous function of degree 2 (two)

Thus as both $M(x,y) = 2xy$ and $N(x,y) = x^2$ are homogeneous functions of degree 2, we conclude that the first order differential equation ($df = 2xydx + x^2dy$) is a homogeneous function of degree 2(two).

$$\text{Now if } M(x,y) = 2xy$$

$$\delta M/\delta y = 2x$$

$$\text{Again if } N(x,y) = x^2$$

$$\delta N/\delta y = 2x$$

Here as $\delta M/\delta y = \delta N/\delta x = 2x$, then the first order homogeneous differential equation ($df = 2xydx + x^2dy$) is an exact differential equation.

Example 7: *Case of Separation of Variables*

Statement: The Case of separation of variables shows that for a given first order differential equation of the form $M(x,y)dx + N(x,y)dy = 0$, if the differential equation can be written in the form of separated variables by assuming say $y = ux$ and therefore $dy = udx + xdu$, then the equation can be solved easily by the method of integration.

Problem: Solve the equation $(x^2 – y^2)\,dx + (xy)\,dy = 0$ as a case of separation of variables.

Suppose $y = ux$, then the given differential equation $(x^2 – y^2)\,dx + (xy)\,dy = 0$ can be written as

$$[x^2 – (ux)^2]dx + [x.ux]dy = 0$$

$$\text{i.e. } [x^2 – (ux)^2]dx + [x.ux][udx + xdu] = 0$$

{already stated $dy = udx + xdu$}

$$\text{i.e. } (x^2 – u^2x^2)dx + u^2x^2dx + ux^3du = 0$$

$$\text{i.e. } x^2dx – u^2x^2dx + u^2x^2dx + ux^3du = 0$$

$$\text{i.e. } x^2 dx + ux^3 du = 0$$

$$\text{i.e. } x^2 (dx + ux\, du) = 0$$

$$\text{i.e. } (dx + ux\, du) = 0$$

$$\text{i.e. } = - dx/x$$

Integrating both sides

$$\text{i.e. } \int u\, du = \int - (dx/x)$$

$$\text{i.e. } u^2/2 = - \log x + C$$

{C is the constant of integration}

Let us insert y =ux, we get u = y/x. Then $u^2/2 = - \log x + C$, can be written as

$$½ (y/x)^2 = -\log x + C$$

Let us assume C =log C', then we can write

$$½ (y/x)^2 = -\log x + \log C'$$

$$\text{i.e. } ½ (y/x)^2 = \log C' - \log x$$

$$½ (y/x)^2 = \log (C'/x)$$

$$\text{i.e. } y^2 = 2x^2 \log (C'/x)$$

The General Formula

Statement: A first-order linear differential equation, can be expressed in the form

$$dy/dx + Py = Q$$

where dy/dx and y must be of the first degree, P and Q are functions of x or P and Q are constants and no product [y (dy/dx)] may occur. The general solution for this equation is given by

$$y = e^{-\int P\, dx} \int e^{\int P\, dx} Q\, dx + c\, e^{-\int P\, dx}$$

The general solution helps us to solve three distinct cases of differential equations:

- Homogeneous Differential Equations
- Non-Homogeneous Differential Equations
- General Cases

Case I: Homogeneous Differential Equation

When Q = 0, and P is a function of x, we have a homogenous differential equation and the solution is given by

$$y = c\, e^{-\int P\, dx}$$

Example 8: Find the solution of the differential equation dy/dx +xy =0.

For the given differential equation dy/dx +xy =0, which is in the form dy/dx +Py = Q

P = x and Q = 0. Thus the solution is obtained as follows:

$$y = c\, e^{-\int x\, dx} = c\, e^{-x^2/2}$$

Case II: When P and Q are constants and we have a non-homogeneous equation with constant coefficients the solution becomes

$$y = e^{-Px} \int e^{Px} Q\, dx + c\, e^{-Px}$$

$$\text{i.e. } y = c\, e^{-Px} + Q/P$$

Example 9: Find the solution of the differential equation dy/dx +3y = 4.

For the given differential equation dy/dx +3y = 4, which is in the form dy/dx +Py = Q

P = 3 and Q = 4. Thus the solution is obtained as follows:

$$y = c\, e^{-3x} + 4/3$$

Case III: When P and Q are functions of x, the general solution is as stated earlier

$$y = e^{-\int P\, dx} \int e^{\int P\, dx} Q\, dx + c\, e^{-\int P\, dx}$$

Example 10: Find the solution of the differential equation dy/dx + (1/x)y = x.

For the given differential equation dy/dx + (1/x)y = x, which is in the form dy/dx +Py = Q

P = 1/x and Q = x. Thus the solution is obtained as follows:

$$y = e^{-\log x} \int e^{\log x}\, x\, dx + c\, e^{-\log x}$$

$$y = x^2/3 + c/x \quad [\text{Note: } e^{\log x} = x]$$

*N.B. Explanations of all the theories and examples have been taken from the following references:

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