Economics Honours (Second Semester)

Mathematical Economics – I

Some Applications of Integral Calculus in Economics

Determination of Total Cost

Let x be the output and the total cost function be

$$f(x) = 100 + 3x + 4x^2 + 5x^3$$

Then the marginal cost is $f'(x) = 3 + 8x + 15x^2$

We can reverse the procedure and if given the marginal cost function, we can determine the total cost function. We know that $f(x) = \int f'(x)\,dx$

$$f(x) = \int (3 + 8x + 15x^2)\,dx = 3x + 4x^2 + 5x^3 + c$$

We can determine the value of c if we have an additional condition. Let it be that f(0) = 100.

Thus,

$$f(0) = 3(0) + 4(0) + 5(0) + c = 100, \text{ i.e. } c = 100.$$

Hence the total cost function is

$$f(x) = 100 + 3x + 4x^2 + 5x^3$$

Example: If the marginal cost of a firm is $MC = 4 + 6q + 30q^2$, find the total cost function given that the fixed cost is Rs 70/-.

Suppose the total cost is given by C = C(q), then MC = dC/dq.

Given $MC = 4 + 6q + 30q^2$, we can write the function as

$$dC/dq = 4 + 6q + 30q^2$$

Here integrating the marginal cost function on both sides gives the total cost function.

$$\text{Thus } \int dC = \int (4 + 6q + 30q^2)\,dq$$

$$\text{Or, } C = 4q + 6q^2/2 + 30q^3/3 + k \text{ [where k is supposed to be a constant]}$$

$$\text{Or, } C(q) = 4q + 3q^2 + 10q^3 + k \text{ [Stated } C = C(q)]$$

We can determine the k if we have an additional condition. Let it be that C(0) = 70.

Thus C(0) = 4(0) + 3(0) +10 (0) + k = 70.

Hence the total cost function is $C(q) = 70 + 3q^2 + 10q^3$

Answer: Total cost = $C(q) = 70 + 3q^2 + 10q^3$

Consumers' Surplus and Producers' Surplus

Suppose for given commodity x and its price p, if the inverse demand function is given by $p = Q(x)$ and the supply function by $p = S(x)$, then from solving these equations simultaneously gives the equilibrium level of the commodity and the equilibrium price. Suppose the determined equilibrium price- quantity combination is (x^*, p^*), then

$$\text{Consumers' Surplus} = CS = \int_0^{x^*} Q(x)\,dx - p^*x^*$$

$$\text{Producers' Surplus} = PS = p^*x^* - \int_0^{x^*} S(x)\,dx$$

Example: Suppose for a given commodity x and its price, the demand equation is $p = 142 – 3x^2$ and the supply equation is $p = 46 + 3x^2$. Find the consumers' surplus and producers' surplus.

At equilibrium, the supply of x is equal to the demand for x. Thus we have

$$142 – 3x^2 = 46 + 3x^2,$$

$$\text{Or, } 6x^2 = 96$$

$$\text{Or, } x^2 = 16$$

$$\text{Or, } x = \pm\sqrt{16}$$

$$\text{Or, } x = \pm 4.$$

As $x \neq -4$, hence $x = x^* = 4$, where x^* is the equilibrium quantity.

Putting $x^* = 4$ in either the demand or supply equation, we obtain the equilibrium price, p^*. We get $p^* = 94$.

By the formula for Consumers' Surplus (CS) , we get

$$CS = \int_0^4 (142 – 3x^2)dx - (94 \text{ x} 4) = \left[142x - \frac{3x^3}{3}\right]_0^4 - (376) = [\,568 – 64] – [0] – 376 = 128.$$

By the formula for Producers' Surplus (PS) , we get

$$PS = (94 \text{ x } 4) - \int_0^4 (46 + 3x^2)dx = (376) - \left[\,46x + \frac{3x^3}{3}\right]_0^4 = (376) – [184 + 64] – 0 = 128$$

Answer: Consumers' Surplus = Rs 128 and Producers' Surplus = Rs 128

Present value of capital

Discounting: In terms of capital values, the Compound interest formula is given as

$$P_t = P_0 (1 + i)^t$$

P_t = value of capital at time t

P_0 = initial value of capital

t = number of years

i = effective rate of interest

The initial capital is compounded annually and the rate of interest is i which is called the effective rate of interest.

Thus the initial value of capital can be written as

$$P_0 = P_t (1 + i)^{-t}$$

where i is the annual rate of interest. This process of finding the value of an investment P_t at some earlier time, say t=0, is called discounting.

For continuous compounding, P_0 will be

$P_0 = P_t e^{-\delta t}$ where δ is the force of interest.

Value of capital asset: Let $y_1, y_2, y_3, \ldots\ldots$ be the expected yield of a capital asset. Then the present value of capital asset will be, assuming continuous compounding,

$$C = \int_0^t y_t e^{-rt} dt$$

where r is the force of interest

If we assume y_t = f (t) = y = constant say for example rent, the C becomes

$$C = \int_0^t y e^{-rt} dt = y \int_0^t e^{-rt} dt = \left[y(- \frac{1}{r} e^{-rt}) \right]_0^t$$

$$= y(- \frac{1}{r} e^{-rt}) - y(- \frac{1}{r})$$

$$= \frac{y}{r}(1 - e^{-rt})$$

Thus the present value of $C = \frac{y}{r}(1 - e^{-rt})$

If $t \to \infty$, then C = y/r.

Solve the following problems

1. Suppose x is the output and the total cost function is given as $f(x) = 100 + 3x + 4x^2 + 5x^3$, find the marginal cost.

2. Given the marginal cost function $f'(x) = 2 + x + x^2$, find the total cost.

3. Given the marginal cost function $f'(x) = 10 - 2x$ and $f(0) = 50$. Find the total cost

4. Given the marginal propensity to consume $dC/dy = f'(y)$, find $C = f(y)$

   a. $f'(y) = 0.680 - 0.0012y$ and $f(0) = 0.128$

   b. $f'(y) = -0.23(1/y_0)$ where y_0 is constant and $f(0) = 117.96$

5. Given the marginal cost $MC = 3q^2 - 4q + 6$ and total fixed cost =8, can we claim that AC is minimum when q =2? Derive the minimum value of AC.

6. Obtain the consumption function when $dy/ds = 1/2$ and minimum consumption is 50 at any level of y.

7. Given the following MPC function, derive the consumption function:
$MPC = C'(Y) = 0.4\,(1/\sqrt{y})$. What is meant by indefinite integral?

8. Obtain the consumer's surplus from the following demand function when p =8 and
$Q = \sqrt{(42 - 0.75p)}$. What is meant by definite integral?

9. If the marginal cost function of a firm is: $MC = 4 + 6x + 30x^2$, find its total cost function given total fixed cost as Rs 200/-.

10. If consumer's demand function is given by $Q = f(P) = \sqrt{(145 - 5P)}$, find the consumer's surplus when market price = 9.

11. Obtain the savings function when $dc/dY = 1/2$ and C =50 at Y=0

12. Suppose $Mu_x = ax^2$. What will be the total utility function of x?

*N.B. Explanations of all the theories, examples and problems have been taken from the following references:

Notes and References:

- Yamane, T. (1962). *Mathematics for economists: An elementary survey* (2nd ed.). Prentice hall of India Private Limited.
- Question papers of previous years
- Doubts expressed by learners