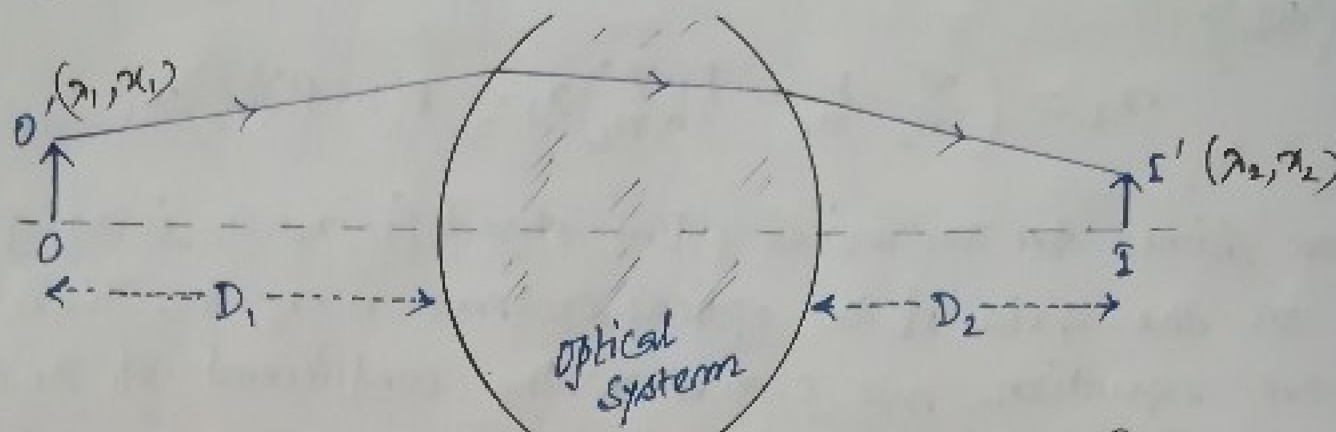
Image formation by an optical system :



Let us consider an object is kept infront of an optical system and the object is in ~~a~~ the air medium. The optical coordinates of O' and I' are (λ_1, x_1) and (λ_2, x_2) respectively. Suppose the system matrix of the optical system is

$$S = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix}$$

So, for the image formation,

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D_2 & 1 \end{pmatrix} \begin{pmatrix} b & -a \\ -d & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -D_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} b & -a \\ bD_2 - d & c - aD_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -D_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b + aD_1 & -a \\ bD_2 - d - cD_1 + aD_1D_2 & c - aD_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{---- (i)}$$

~~Now for an axial~~

from the above

$$\lambda_2 = (b + aD_1)\lambda_1 - ax_1 \quad \text{------ (ii)}$$

and $$x_2 = (bD_2 - d - cD_1 + aD_1D_2)\lambda_1 + (c - aD_2)x_1 \quad \text{-- (iii)}$$

Now for an axial point object i.e for $x_1 = 0$, the image point must ~~lie~~ lie on the axis i.e $x_2 = 0$. So, the coefficient of λ_1 must be vanished.

$$\therefore \; bD_2 - d - cD_1 + aD_1D_2 = 0 \quad \text{-------- (A)}$$

Therefore corresponding image plane will be

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b+aD_1 & -a \\ 0 & c-aD_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \cdots\cdots\cdots \text{(iv)}$$

Since this operation does not change the characteristics of light, so the

$$(b+aD_1)(c-aD_2) = 1$$

$$\therefore (b+aD_1) = \frac{1}{c-aD_2} \quad \cdots\cdots\cdots\cdots\cdots (B)$$

Again, from the above matrix formulation

$$\lambda_2 = (b+aD_1)\lambda_1 - ax_1$$

$$x_2 = (c-aD_2)x_1$$

$\therefore$ Magnification $M = \frac{x_2}{x_1} = c - aD_2$

$\therefore$ from equation (v), $b + aD_1 = \frac{1}{M}$

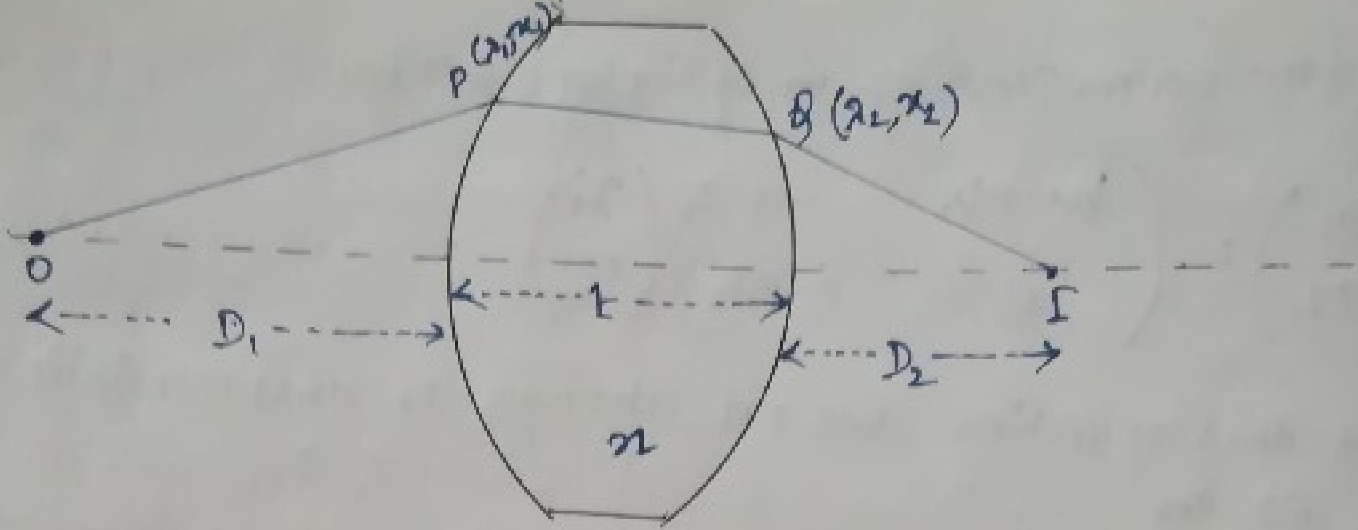
Thus, the above matrix formulation (iv) can be written as

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{M} & -a \\ 0 & M \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$$

This is the matrix formulation for the image formation by an optical system.

System matrix for a thick lens; hence thin lens formula and thick lens formula.

Let us consider a thick lens of thickness t having refractive index n. Let R_1 and R_2 are the radius of curvatures of first and second surface of the lens. Let a point object is placed in air at a distance D_1 and the image formed by the lens on another side at D_2. A ray strikes the lens first surface at $P(\lambda_1, x_1)$ and emerges from the second surface from $Q(\lambda_2, x_2)$.



So, $$\begin{pmatrix}\lambda_2\\ \alpha_2\end{pmatrix} = \begin{pmatrix}1 & -P_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ t/n & 1\end{pmatrix}\begin{pmatrix}1 & -P_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\ \alpha_1\end{pmatrix}$$

Here the system matrix

$$S = \begin{pmatrix}b & -a\\ -d & c\end{pmatrix} = \begin{pmatrix}1 & -P_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ t/n & 1\end{pmatrix}\begin{pmatrix}1 & -P_1\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1-P_2 t/n & -P_2\\ t/n & 1\end{pmatrix}\begin{pmatrix}1 & -P_1\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1-P_2 t/n & -P_1+P_1P_2 t/n - P_2\\ t/n & 1-P_1 t/n\end{pmatrix} \quad \text{--- (i)}$$

Now for a thin lens, $t \to 0$

So, the system matrix for thin lens will be

$$S = \begin{pmatrix}b & -a\\ -d & c\end{pmatrix} = \begin{pmatrix}1 & -(P_1+P_2)\\ 0 & 1\end{pmatrix} \quad \text{------ (ii)}$$

Comparing the coefficients

$a = P_1+P_2$, $b=1$, $c=1$, $d=0$.

Now from the boundary condition of image formation, we can use equation (A).

$$bD_2 + aD_1D_2 - cD_1 - d = 0 \quad \text{------ (A)}$$

Now substituting above values of a, b, c and d, we get

$$D_2 + (P_1+P_2)D_1D_2 - D_1 = 0 \quad \text{------ (iii)}$$

Now, $P_1 = \frac{m-1}{R_1}$ and $P_2 = \frac{1-m}{R_2}$

$$= -\frac{m-1}{R_2}$$

$$\therefore (P_1 + P_2) = (m-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f}$$

So, the equation (iii) can be written as

$$D_2 + \frac{1}{f} D_1 D_2 - D_1 = 0$$

$$D_1 - D_2 = \frac{1}{f} D_1 D_2$$

$$\therefore \frac{1}{D_2} - \frac{1}{D_1} = \frac{1}{f}$$

which is well known thin lens formula.

For thick lens the values of the coefficients are

$a = P_1 + P_2 - \frac{t}{m} P_1 P_2$, $b = 1 - P_2 t/m$, $c = 1 - P_1 t/m$, $d = -t/m$.

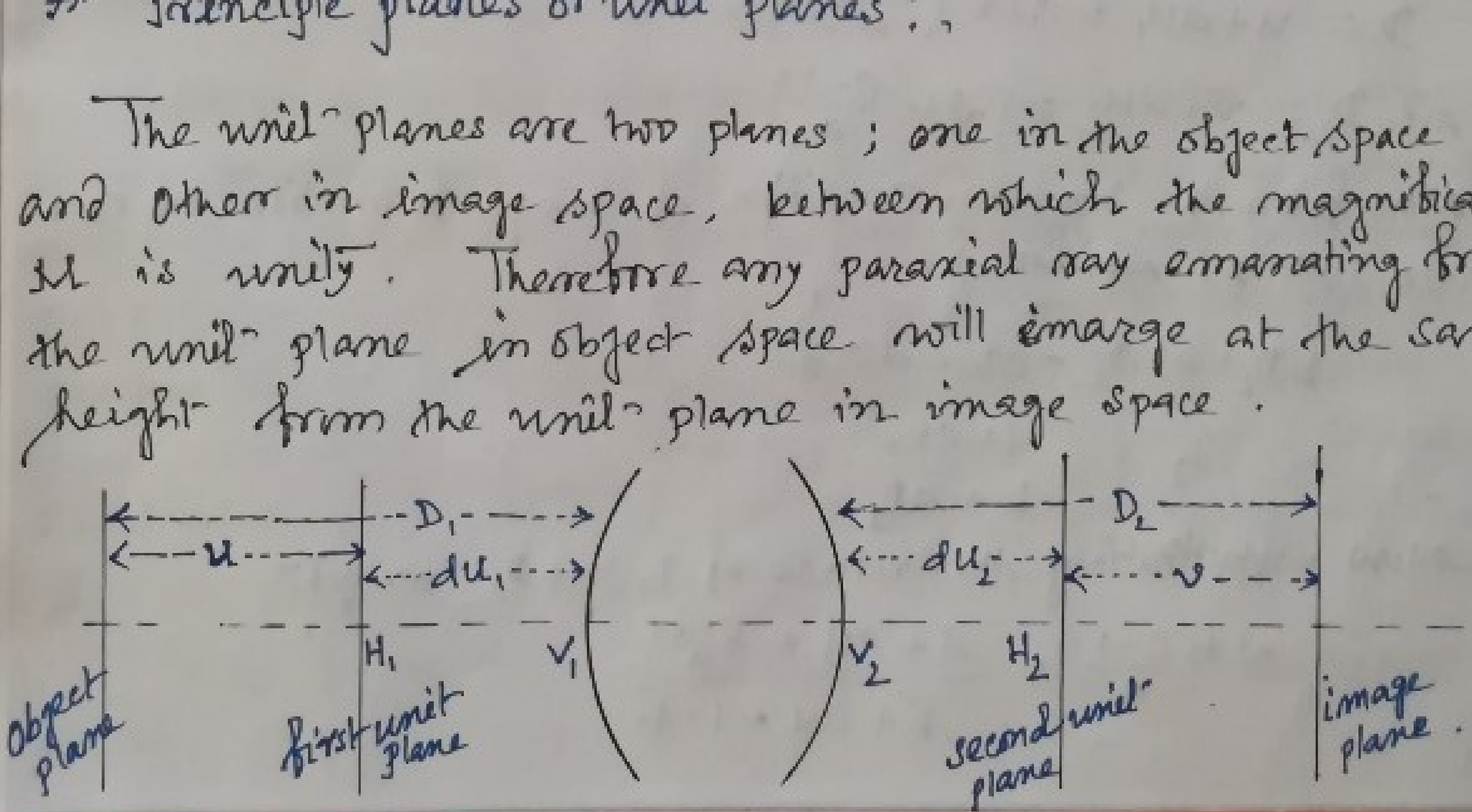
Substituting these values in equation A

$$\left(1 - P_2 \frac{t}{m}\right) D_2 + \left(P_1 + P_2 - \frac{t}{m} P_1 P_2\right) D_1 D_2 - \left(1 - P_1 t/m\right) D_1 + t/m = 0$$

The above one is a complex equation. Actually we shall use the concept of unit planes for deriving thick lens formula. This is done in the following section.

Principle planes or unit planes :

The unit planes are two planes ; one in the object space and other in image space, between which the magnification M is unity. Therefore any paraxial ray emanating from the unit plane in object space will emarge at the same height from the unit plane in image space.



Let du_1 and du_2 are the distances of first and second unit plane respectively from the refracting surfaces then from expression (B)

$$b + adu_1 = \frac{1}{c - adu_2} = 1 \qquad [\because \mu = 1]$$

So.
$$du_1 = V_1H_1 = \frac{1-b}{a}$$
$$du_2 = V_2H_2 = \frac{c-1}{a}$$

If ~~dat~~ n_1 and n_2 be the refractive indices of the media ~~part~~ in left and right of the refracting surface, then

$$du_1 = V_1H_1 = n_1\left(\frac{1-b}{a}\right)$$
$$du_2 = V_2H_2 = n_2\left(\frac{c-1}{a}\right)$$

* Thick lens formula using the concept of unit planes:

Let us consider first and second ~~unt~~ unit planes which are at a distance du_1 and du_2 from the refracting surfaces. Let u be the distance of the object plane from the first unit plane and v be the corresponding distance of image plane from 2nd unit plane. Then

$$D_1 = u + du_1 = u + \left(\frac{1-b}{a}\right)$$
and $$D_2 = v + du_2 = v + \left(\frac{c-1}{a}\right)$$

Now from the boundary condition of image formation we can use eqn (A) which is

$$bD_2 + aD_1D_2 - cD_1 - d = 0$$
$$\therefore D_2 = \frac{d + cD_1}{b + aD_1}$$

Now substituting the values of D_1 and D_2 we get

$$v + \frac{c-1}{a} = \frac{d + cu + \frac{c - bc}{a}}{\cancel{b} + au + 1 - \cancel{b}}$$

or, $v + \frac{c-1}{a} = \frac{ad + acu + c - bc}{a(au+1)}$

or, $v = \frac{ad + acu + c - bc}{a(au+1)} - \frac{c-1}{a}$

$= \frac{ad + \cancel{acu} + \cancel{c} - bc - \cancel{acu} - \cancel{c} + au + 1}{a(au+1)}$

$= \frac{ad - bc + au + 1}{a(au+1)}$

Now, $|S| = \begin{vmatrix} b & -a \\ -d & c \end{vmatrix} = bc - ad = 1$

$\therefore v = \frac{-1 + au + 1}{a(au+1)}$

$= \frac{u}{au+1}$

$\therefore \frac{1}{v} = a + \frac{1}{u}$

$\therefore \frac{1}{v} - \frac{1}{u} = a$

which is well known thick lens formula

Deduce an expression for the positions of nodal planes :-

Nodal points are two points on the axis which have a relative angular magnification of unity. That means if a ray strikes the first nodal point at an angle α, then it will emerge at same angle from the second nodal point. Planes through that points and ~~perp~~ normal to the axis is called first and second nodal plane respectively. Let N_1 and N_2 denotes the first and second nodal points and dn_1 and dn_2 be the corresponding distances from the refracting surfaces.

Now for the formation of image of an object by an optical system

$$\begin{pmatrix} x_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} b+aD_1 & a \\ 0 & c-aD_2 \end{pmatrix} \begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix}$$

Now from the definition of nodal points

$$\frac{V_2}{V_1} = 1$$

or, $\dfrac{nV_2}{nV_1} = 1 \qquad \therefore x_2 = x_1$

Again for axial point object $\alpha_1 = 0$

$$\therefore \cancel{b+aD_1}\; x_2 = (b + a\,dn_1)\, x_1$$

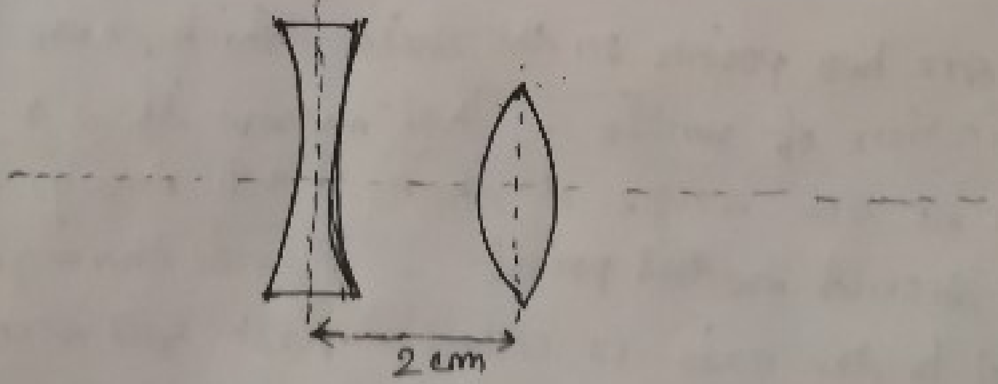
$$\therefore b + a\,dn_1 = 1$$

$$dn_1 = \frac{1-b}{a}$$

Similarly $dn_2 = \dfrac{c-1}{a}$

So, when both media on either side of an optical system have same refractive index ie $n_1 = n_2 = n$, the nodal planes coinsides with the unit planes.

* A lens of -20 cm focal length is placed 2 cm in front of a lens of $+4$ cm focal length. Find the system matrix of the combination and check by determinant.



$f_1 = -20$ cm so, $P_1 = \dfrac{100}{-20} = -5\ m^{-1}$

And $f_2 = +4$ cm $\therefore P_2 = \dfrac{100}{4} = +25\ m^{-1}$

Now the system matrix of the combination

$$S = \begin{pmatrix} 1 & -25 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{100} & 1 \end{pmatrix} \begin{pmatrix} 1 & +5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -25 \\ \frac{1}{50} & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -45/2 \\ 1/50 & 11/10 \end{pmatrix}$$

Now, $|S| = \begin{vmatrix} 1/2 & -45/2 \\ 1/50 & 11/10 \end{vmatrix} = \frac{1}{2} \times \frac{11}{10} + \frac{45}{100} = \frac{11}{20} + \frac{9}{20} = 1$

* Two thin lenses have a combined power of +10D . If they become separated by 20 cm their ~~equa~~ equivalent power decreases to +6.25 D . What are the power of two lenses.

⇒ When two lenses are in contact then the system matrix of the combination is

$$S = \begin{pmatrix} 1 & -(P_1+P_2) \\ 0 & 1 \end{pmatrix}$$

Now $(P_1 + P_2) = 10$ - - - - - - - - (i)

When they are separated then

$$S' = \begin{pmatrix} 1 - \frac{d_0}{n} P_2 & -(P_1 + P_2 - \frac{d_0}{n} P_1 P_2) \\ \frac{d_0}{n} & 1 - \frac{d_0}{n} P_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{1}{5} P_2 & -(P_1 + P_2 - \frac{1}{5} P_1 P_2) \\ 1/5 & 1 - \frac{1}{5} P_1 \end{pmatrix} \qquad \left[d_0 = \frac{20}{100} = \frac{1}{5} m \right]$$

Then power of the combination

$$6.25 = P_1 + P_2 - \frac{1}{5} P_1 P_2$$

$$= 10 - \frac{1}{5} P_1 P_2 \qquad [\text{Using eq}^n \text{(i)}]$$

$\therefore \frac{1}{5} P_1 P_2 = 3.75$

$PP_2 = 5 \times 3.75$

Now $(P_1 - P_2)^2 = (P_1 + P_2)^2 - 4 P_1 P_2$

$= 10^2 - 4 \times 5 \times 3.75$

$= 100 - 75$

$= 25$

$\therefore P_1 - P_2 = \pm 5$

So,

$P_1 + P_2 = 10$

$P_1 - P_2 = 5$

$2P_1 = 15$

$P_1 = 15/2$

$\therefore P_2 = 10 - 15/2$

$= 5/2$

And

$P_1 + P_2 = 10$

$P_1 - P_2 = -5$

$2P_1 = 5$

$P_1 = 5/2$

$\therefore P_2 = 10 - 5/2 = 15/2$

So, the power of the two lenses are $15/2$ D, $5/2$ D.