Matrix Method

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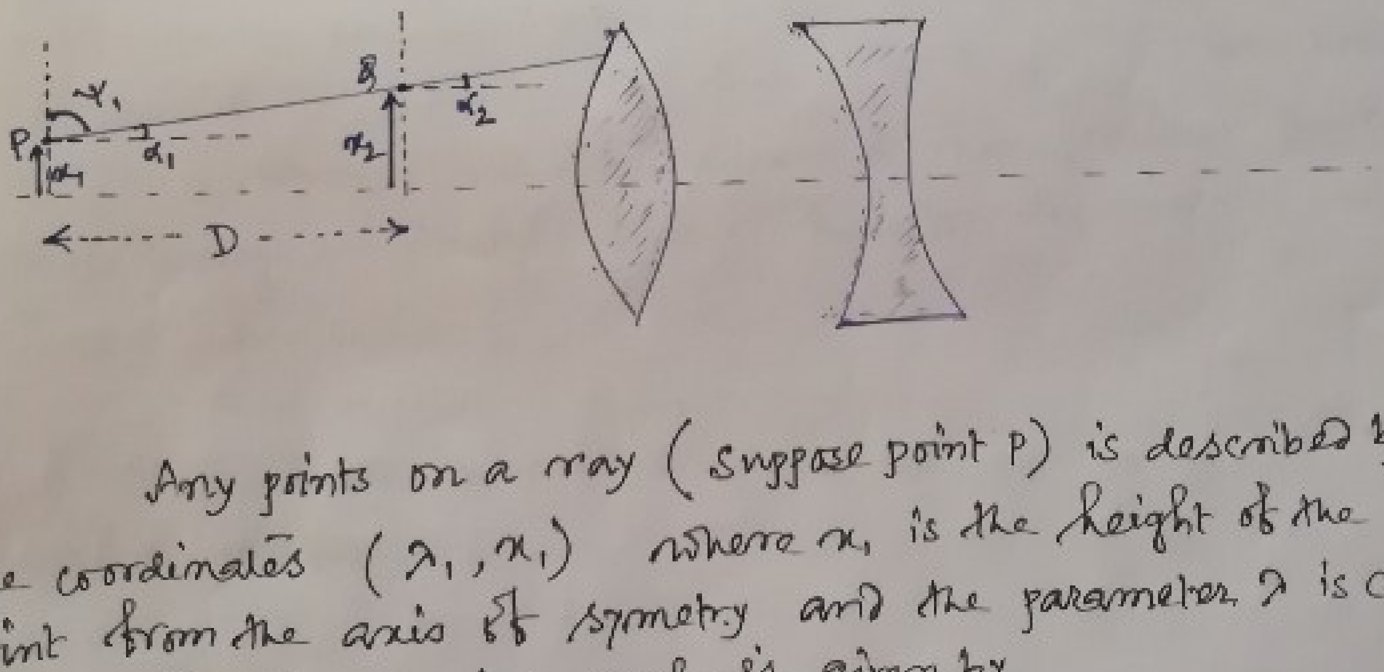
When a ray propagates through different optical systems one has to calculate step by step the position of the image due to each surface and this image will act as an object for the next surface to obtain the final image. Such a step by step calculation becomes complicated with the increase of number of optical elements of the system. In these case we can use the matrix method. In particular, this method of analysis can be performed directly by using modern computers.

Here we will consider only paraxial rays, as non-paraxial rays leads to some defect of image, known as aberration.



Any points on a ray (suppose point P) is described by the coordinates (λ_1, x_1) where x_1 is the height of the point from the axis of symmetry and the parameter λ is called optical direction cosine and is given by

$$\lambda_1 = n_1 \cos\alpha_1 = n_1 \sin\alpha_1$$

[where n_1 is the refractive index of the ~~pointy~~ medium where point P is situated]

In paraxial approximation α_1 is small

so, $\lambda_1 = n_1 \alpha_1$

so, the optical coordinate of point P is (λ_1, x_1) and in matrix method this coordinates are written in the form of column matrix as $\begin{bmatrix} \lambda_1 \\ x_1 \end{bmatrix}$. Similarly, for point Q the coordinates can be written as $\begin{bmatrix} \lambda_2 \\ x_2 \end{bmatrix}$.

Now when a ray propagates then translation and refraction occurs. When a ray of light passes through refracting surfaces (e.g. a combination of lenses) at each surface the direction of the ray changes but the height does not. This is called refraction. Within each interval between surfaces the height of the ray changes but the direction does not, this is called translation.

Thus we need two operators one for refraction process and other for translation process to fully describe the ray as it progresses through surfaces after surfaces.

Derivation for translational matrix :-

Suppose a ray is travelling in a homogenious medium of refractive index n_1 from point P to Q. The optical coordinates of the ray at P is (λ_1, x_1) and that at Q is (λ_2, x_2) as shown in the above figure. As the medium is homogenious the ray travells along a straight line. Therefore the angle α_1 and α_2 are equal. Hence

$$n_2\alpha_2 = n_1\alpha_1$$

$$\therefore \lambda_2 = \lambda_1 \quad \text{------------} \quad (i)$$

Now from the fig.

$$x_2 = x_1 + D\tan\alpha_1$$

Since we are considering paraxial rays.

$$\text{So,} \quad x_2 = x_1 + D\alpha_1$$

$$\therefore x_2 = x_1 + D\frac{\lambda_1}{n_1} \quad \text{------------} \quad (ii)$$

Eqn (i) & (ii) may be combined in to the following matrix equation

$$\begin{bmatrix} \lambda_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ D/n_1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ x_1 \end{bmatrix}$$

from the above we see that translation matrix is an operator

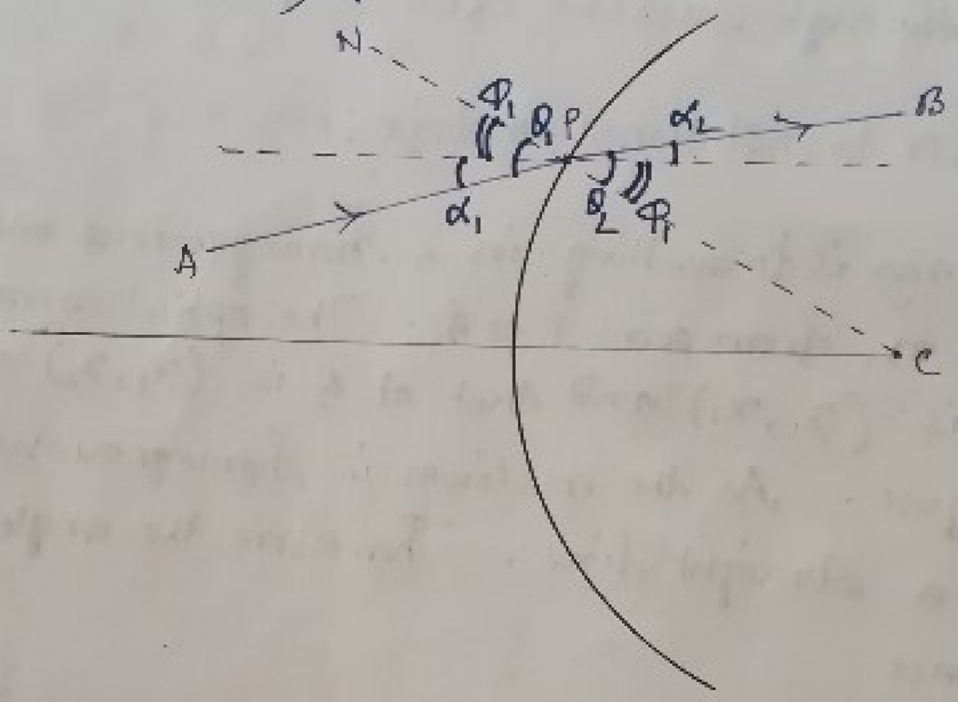
On the ray which moves the point P to Q and can be represented by a (2×2) matrix.

$$T = \begin{bmatrix} 1 & 0 \\ D/n_1 & 1 \end{bmatrix} \quad \text{------------ (a)}$$

Note that determinant of T is 1.

So, the translation matrix T transform the ray $\begin{bmatrix} \lambda_1 \\ x_1 \end{bmatrix}$ in to the ray $\begin{bmatrix} \lambda_2 \\ x_2 \end{bmatrix}$ during the translation through a distance D in homogenious medium.

Refraction matrix :-



Let us consider a ray AP intersecting a spherical surface of radius of curvature R, separating two media of refractive indices n_1 and n_2. If θ_1 and θ_2 be the angle of incidences and angle of refraction then according to Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

For paraxial approximation, θ_1 and θ_2 both are small

so, $n_1\theta_1 = n_2\theta_2$

$\therefore n_1(\phi_1+\alpha_1) = n_2(\phi_1+\alpha_2)$

Now, $\phi_1 = \frac{x_1}{R}$

$\therefore \quad n_2\alpha_2 = n_1\alpha_1 - \frac{(n_2-n_1)x_1}{R}$

$$\lambda_2 = \lambda_1 - P x_1 \quad \text{------} \text{(i)}$$

where $P = \frac{n_2 - n_1}{R}$ is called power of the refracting surface.

Now since height of the ray just before and after refraction is same. so,

$$x_2 = x_1 \quad \text{------} \text{(ii)}$$

so, the above pair of equation can be written in matrix form as

$$\begin{bmatrix} \lambda_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ x_1 \end{bmatrix}$$

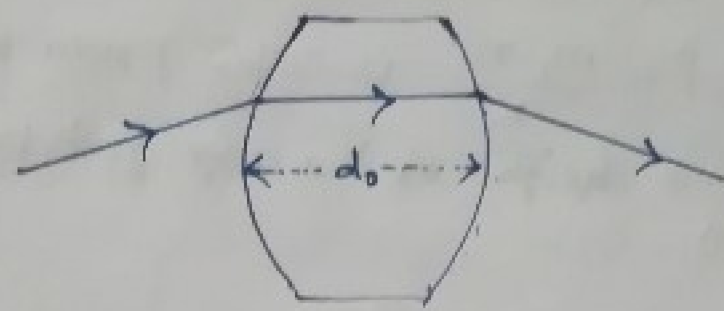
So, a (2x2) matrix which transform the incident ray $\begin{bmatrix} \lambda_1 \\ x_1 \end{bmatrix}$ in to refracted ray $\begin{bmatrix} \lambda_2 \\ x_2 \end{bmatrix}$ during refraction at the surface is called refraction matrix.

Hence refraction matrix $\mathcal{R} = \begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix}$

Now. $\det \mathcal{R} = \begin{vmatrix} 1 & -P \\ 0 & 1 \end{vmatrix} = 1$

Significance :- Here we note that for both the translation matrix and refraction matrix we get the determinant is unity. The significance of this unit determinant is that here these operations (translation, refraction or their combination) do not change the characteristics of light. They only change the position and orientation of the ray of light.

System matrix:-



When a ray passes through a thick lens refraction occurs twice at each surface and translation occurs once between surfaces. Hence in order to describe the propagation of the ray we need two refraction matrix R_1 and R_2 and one translation matrix T_1. Multiplication of these matrices written from right to left we will get the desired system matrix S. Thus

$$S = \mathcal{R}_2 T_1 \mathcal{R}_1 \quad \text{-------- (i)}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the refraction matrices at the first and second surface of the lens. So,

$$S = \begin{bmatrix} 1 & -P_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d_0}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix}$$

where n is the refractive index of the material of lens.

$$= \begin{bmatrix} 1 - \frac{d_0}{n} P_2 & -P_2 \\ \frac{d_0}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -P_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{d_0}{n} P_2 & -\left(P_1 + P_2 - \frac{d_0}{n} P_1 P_2\right) \\ \frac{d_0}{n} & 1 - \frac{d_0}{n} P_1 \end{bmatrix} \quad \text{--- (ii)}$$

This is system matrix for thick lens, where P_1 and P_2 are the power of first and second surface of the lens.

$$P_1 = \frac{n-1}{R_1} \quad \text{and} \quad P_2 = \frac{1-n}{R_2}$$

where R_1 and R_2 are radius of curvature of two surfaces.

For thin lens $d_0 = 0$

Therefore

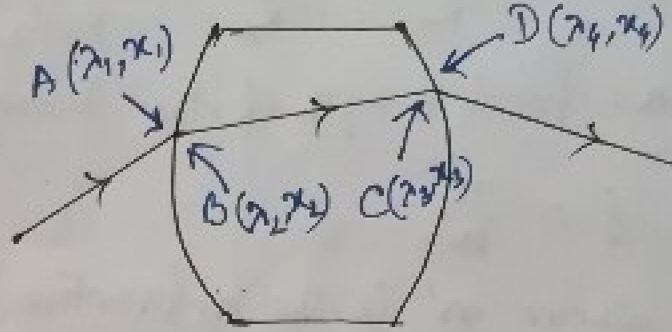
$$S = \begin{bmatrix} 1 & -(P_1+P_2) \\ 0 & 1 \end{bmatrix}$$

<u>Note</u>:. For more confidential of eq(i), we can see the following.

$$\begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix} = \mathcal{R}_1 \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_3 \\ \alpha_3 \end{pmatrix} = T_1 \begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_4 \\ \alpha_4 \end{pmatrix} = \mathcal{R}_2 \begin{pmatrix} \lambda_3 \\ \alpha_3 \end{pmatrix}$$



combining the above equations for final ray

$$\begin{pmatrix} \lambda_4 \\ \alpha_4 \end{pmatrix} = \mathcal{R}_2 \begin{pmatrix} \lambda_3 \\ \alpha_3 \end{pmatrix} = \mathcal{R}_2 T_1 \begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix} = \mathcal{R}_2 T_1 \mathcal{R}_1 \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \lambda_4 \\ \alpha_4 \end{pmatrix} = S \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

where S is system matrix ; can be written as

$$S = \mathcal{R}_2 T_1 \mathcal{R}_1$$

If the system consist of two such lenses then the system matrix can be written as

$$S = \mathcal{R}_4 T_3 \mathcal{R}_3 T_2 \mathcal{R}_2 T_1 \mathcal{R}_1$$

In general any optical system, however complex, can be described as

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix}$$

where the elements a, b, c, d are known as Gaussian const. and negative sign in two elements have been chosen for convenience. The Gaussian const represent various parameter of the system. constant a is equivalent power. It's resiprocal is the equivalent focal length. constant b and d are related with the magnification of the system. Const. c is related with the distance from the right hand vertex to the second principle plane of the system.

Lens maker equation :-

Equivalent power of the lens

$$a = P_1 + P_2 - \frac{d_0}{n} P_1 P_2$$

$$= \frac{n-1}{R_1} + \frac{1-n}{R_2} + \frac{(n-1)^2}{R_1 R_2} \cdot \frac{d_0}{n}$$

So, the focal length of the lens is given by

$$f = \frac{n'}{a}$$

where n' is the refractive index of the medium behind the lens.

For the air medium

$$f = \frac{1}{a}$$

ie $\frac{1}{f} = a = P_1 + P_2 - \frac{d_0}{n} P_1 P_2$

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d_0}{n R_1 R_2}$$

This is the lens maker equation for thick lens.

For thin lens, the eqn can be obtained by putting $d_0 = 0$

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

So, this is lens maker equation for thin lens.

* Find the system matrix for a bio convex lens of focal length 20 cm, thickness 3 cm and refractive index of material of lens is 1.5.

Ans. $R_1 = +40$ cm and $R_2 = -40$ cm.

$$\therefore P_1 = \frac{1.5-1}{0.4} = 2.5\,m^{-1} \qquad P_2 = \frac{1-1.5}{-0.4} = 2.5\,m^{-1}$$

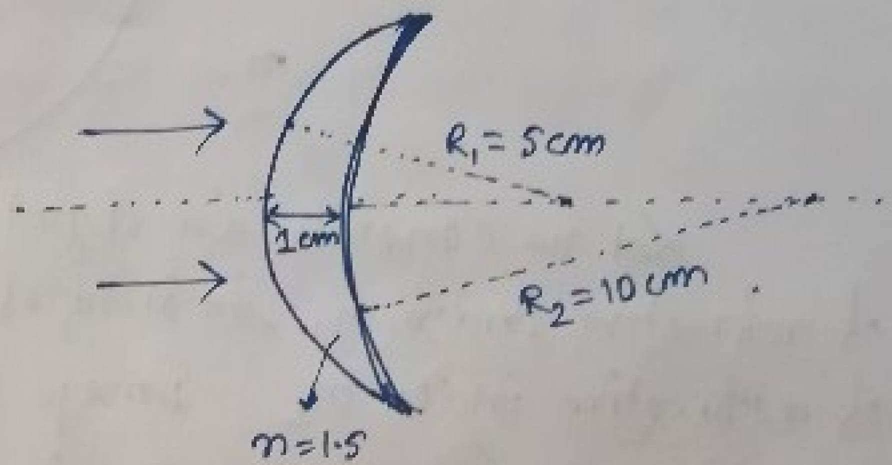
Thus both ~~the~~ surfaces have equal power.

So, the system matrix for the lens is

$$S = \begin{bmatrix} 1 - \frac{0.03}{1.5}\times 2.5 & -\left(2.5 + 2.5 - \frac{0.03}{1.5}\times 2.5\times 2.5\right) \\ \frac{0.03}{1.5} & 1 - \frac{0.03}{1.5}\times 2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.95 & -4.875 \\ 0.02 & 0.95 \end{bmatrix}$$

* Determine the system matrix of the lens shown beside.



Here, $R_1 = +5$ cm
$= 0.05$ m

$\therefore P_1 = \frac{1.5 - 1}{0.05} = 10$ D

And, $R_2 = +10$ cm $= 0.1$ m

$\therefore P_2 = \frac{1 - 1.5}{0.1} = -5$D, $d_0 = 0.01$ m, $n = 1.5$

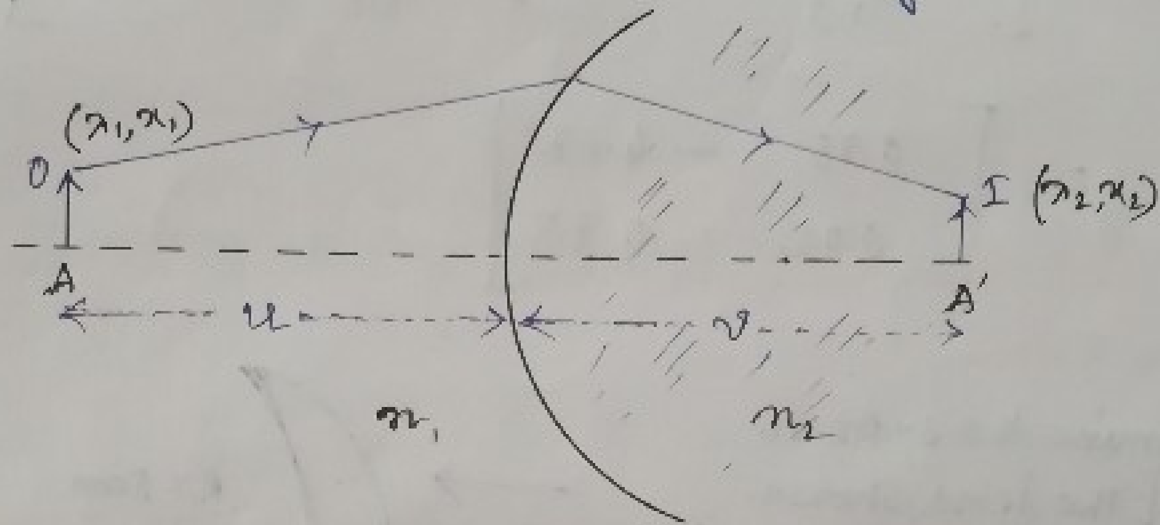
Now, we know that the system matrix

$$S = \begin{bmatrix} 1 - \frac{d_0}{n} P_2 & -\left(P_1 + P_2 - \frac{d_0}{n} P_1 P_2\right) \\ \frac{d_0}{n} & 1 - \frac{d_0}{n} P_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{0.01}{1.5}\times 5 & -\left(10 - 5 + \frac{0.01}{1.5}\times 10\times 5\right) \\ \frac{0.01}{1.5} & 1 - \frac{0.01}{1.5}\times 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{0.1}{3} & -\left(5 + \frac{1}{3}\right) \\ \frac{0.1}{15} & 1 - \frac{1}{15} \end{bmatrix} = \begin{bmatrix} \frac{3.1}{3} & -16/3 \\ \frac{0.1}{15} & \frac{14}{15} \end{bmatrix}$$

Image formed by a spherical refracting surface :-



Let us consider an object OA is kept in a medium of refractive index n_1 in front of a spherical refracting surface of refractive index n_2. Image formed by the spherical refracting surface is shown in the fig. above. Let the optical coordinates of O and I are (λ_1, x_1) and (λ_2, x_2) respectively. Applying matrix method we can write

$$\begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix} = S \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v/n_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -P \\ v/n_2 & 1 - Pv/n_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + P\frac{u}{n_1} & -P \\ \frac{v}{n_2} - \frac{u}{n_1} + \frac{Puv}{n_1 n_2} & 1 - Pv/n_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix} \quad \cdots\cdots (i)$$

So, from the above we can say

$$x_2 = \left(\frac{v}{n_2} - \frac{u}{n_1} + \frac{Puv}{n_1 n_2}\right)\lambda_1 + \left(1 - P\,v/n_2\right)x_1$$

Now since for an axial point object (ie $x_1 = 0$) image point must lie on the axis of the optical system ie $x_2 = 0$. Then from the above equation we can say the coefficient of λ_1 should vanished.

$$\therefore \frac{v}{n_2} - \frac{u}{n_1} + \frac{Puv}{n_1 n_2} = 0$$

$$\text{or. } \frac{u}{n_1} - \frac{v}{n_2} = \frac{Puv}{n_1 n_2}$$

Now multiplying both sides by $\frac{n_1 n_2}{uv}$, we get

$$\frac{n_2}{v} - \frac{n_1}{u} = P = \frac{n_2 - n_1}{R} \quad \text{------- (ii)}$$

So, for the image formation

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + P\frac{u}{n_1} & -P \\ 0 & 1 - P\frac{v}{n_2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{------ (iii)}$$

From the above

$$x_2 = \left(1 - P\frac{v}{n_2}\right)x_1$$

$\therefore$ Magnification

$$m = \frac{x_2}{x_1} = 1 - P\frac{v}{n_2}$$

$$= 1 - \left[\left(\frac{n_2}{v} - \frac{n_1}{u}\right)\frac{v}{n_2}\right] \quad \text{[using eq}^n\text{(ii)]}$$

$$= 1 - 1 + \frac{n_1}{n_2}\cdot\frac{v}{u}$$

$$\therefore m = \frac{n_1}{n_2}\cdot\frac{v}{u}.$$