CLASSICAL DYNAMICS

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Small amplitude oscillations

Part-I

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Introduction

In many situations, a dynamical system is found to be apparently at rest or in some steady state condition. When the system is given a small perturbation, it either settles in an oscillation about its original position of rest (or equilibrium)or it moves away from original position.

When the system is displaced slightly from its “stable” equilibrium position, it undergoes oscillation. The cause of oscillation is the restoring forces which are called into play. Restoring forces can do both positive and negative work. When the work done is positive, the restoring forces change the potential energy into kinetic energy and when the work done is negative, they change kinetic energy back into potential energy. As long as the amplitude of oscillation is small enough, the restoring force is proportional to the displacement ($F=-kx$) and the motion describes an amazingly simple and generic character. The system behaves like a set of independent one-dimensional oscillators. Such oscillators are called linear oscillators. The generalized coordinates describing each oscillator are called normal modes. For linear oscillators, the oscillation frequencies are independent of the amplitude of oscillation. Oscillator motion can be damped in the presence of resistive forces. Resistive forces extract energy from the oscillator. For low velocities, the resistive forces are proportional to velocity. Oscillators, whether damped or undamped, can be driven by external agencies which continuously supply energy to the oscillator to keep it oscillating. Such oscillators are known as “forced” or “driven” oscillators. Driven oscillators can cause amplitude of oscillation to become very large when the driving frequency matches the natural frequency of oscillation. This is known as the phenomenon of resonance.

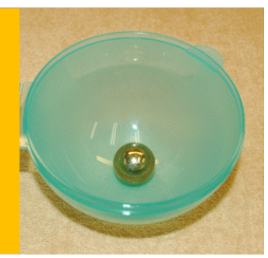
Equilibrium and stability

The state of equilibrium is a defined as one in which the total amount of external force or torque (internal or external) cancel out for some configuration of the system and unless the system is perturbed by an external agency, it stays indefinitely in that state.

Alternatively, it is an state in which the **time derivatives of physical variables** (i.e., observables) **are zero**.

•If the total force on a rigid body is zero then the body shows translational equilibrium as the linear momentum remains unchanged despite the change in time.
•If the total torque on a rigid body is zero then the body shows rotational equilibrium as the angular momentum does not change with time.

Static equilibrium:
It is a state of zero kinetic energy that continues indefinitely and the immediate surrounding of the system does not change with time.
Example: a ball in side a bowl.



Dynamic equilibrium:
It is also a state of zero kinetic energy but the immediate surrounding of the system change with time such that it exerts a balancing force on the system.
Example: a ball static on the head of a fountain.

Equilibrium and stability

Stable, Unstable and neutral equilibrium

A system is said to be in ***stable equilibrium***, if a small displacement of the system from the rest position (by giving a little energy to it) results in a small bounded motion about the equilibrium position. In case, small displacement of the system from the equilibrium position results in an unbounded motion, it is in an ***unstable equilibrium***. Further, if the system on displacement has no tendency to move about or away the equilibrium position, it is said to be in ***neutral equilibrium***. An example of stable equilibrium is a pendulum in the rest position and that of an unstable equilibrium is an egg standing on one end. A coin placed flat anywhere on a table is in neutral equilibrium.

- Stable equilibrium exists if the net force is zero, and small changes in the system would cause an increase in potential energy.

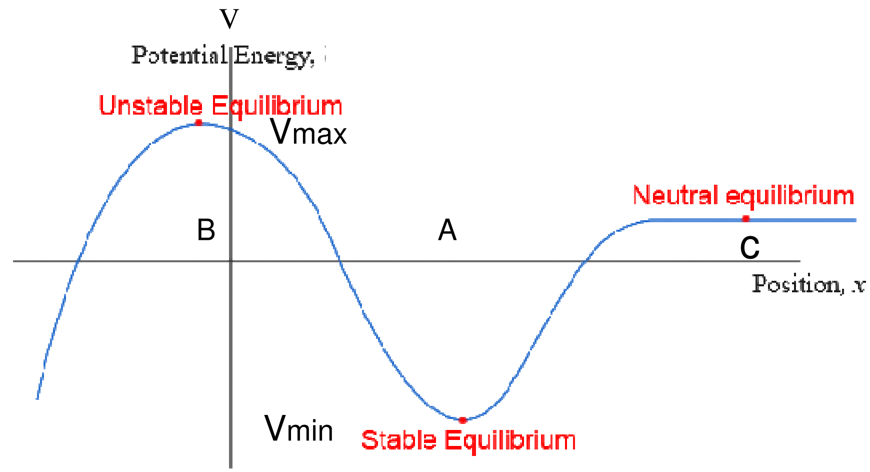
- Unstable equilibrium exists if the net force is zero, and small changes in the system would cause a decrease in potential energy.

- Neutral equilibrium exists if the net force is zero, and small changes in the system have no effect on the potential energy.

Equilibrium and stability

Potential Energy vs. Position Graph

For a conservative system potential energy is a function of position only.
We now know that the (negative of) slope of a potential energy vs. position graph is force.
Equilibrium occurs where the force is zero.
Thus the equilibrium positions are the positions where the slope of the potential energy vs. position curve is zero.



Let V(x) be the potential energy function.
A small deviation from the equilibrium position A (Vmin) would amount to an increase in the potential energy V and as a result the kinetic energy T would decrease, the system being conservative. So the velocity would decrease and the system would eventually come to rest, implying small and bounded motion about A. Such equilibrium is called **stable equilibrium**.

If however a small deviation occurs from position B (Vmax), there is a decrease in V and a consequential increase in the kinetic energy T and hence in velocity. This corresponds to unstable motion and the position of Vmax is called as the position of **unstable equilibrium**.

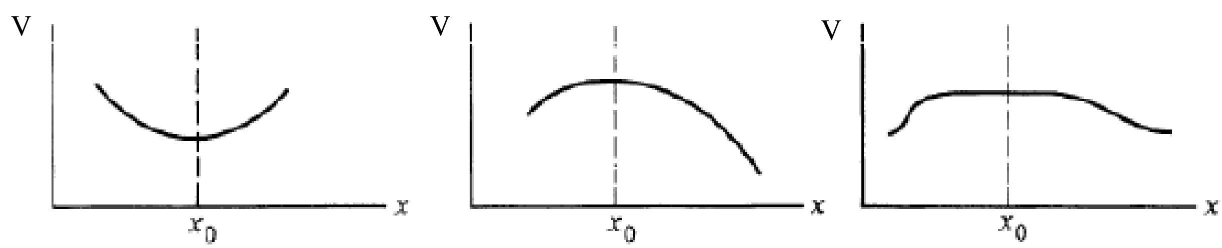
If a small change in position is made from the position C there is no change in potential energy. This is the position of **neutral equilibrium**.

Equilibrium and stability

The result F= -dV/dx is useful not only for computing the force but also for visualizing the stability of a system from the potential energy plot. Suppose there is a force on the particle is F = -dV/dx, and the system is in equilibrium,, i.e., dV/dx= 0. If this occurs at a minimum of V it is a stable equilibrium whereas if it is at a maximum of V, the equilibrium is unstable.

Say, dV/dx= 0 occurs at some point x_0. To find the stability one needs to examine d^2V/dx^2 at x_0. If the second derivative is positive, the equilibrium is stable; if it is negative, the system is unstable. If $d^2V/dx^2 = 0$, one must look at higher derivatives. If all derivatives vanish so that V is constant in a region about x_0, the system is said to be in a condition of neutral equilibrium.

To test for stability we must determine whether V has a minimum or a maximum at x_0.



$\frac{d^2V}{dx^2} > 0$ Stable

$\frac{d^2V}{dx^2} < 0$ Unstable

$\frac{d^2V}{dx^2} = 0$ Neutral

Potential energy and equilibrium

Potential energy and equilibrium:

In order to understand the general theory of oscillations, it is essential to know about the potential energy at the equilibrium configuration. Let us consider a conservative system in which the potential energy is a function of position only. Let the system be specified by n generalized coordinates $q_1, q_2, \ldots, q_n$, not involving time explicitly. For such a system, the potential energy is given by

$$V = V(q_1, q_2, \ldots, q_n)$$

and the generalized forces are given by

$$G_k = -\frac{\partial V}{\partial q_k} \quad \text{where } k = 1, 2, \ldots, n$$

The system is said to be in equilibrium, if the generalized forces acting on the system are equal to zero, *i.e.*,

$$G_k = -\left[\frac{\partial V}{\partial q_k}\right]_0 = 0$$

Thus the potential energy has an extremum at the equilibrium configuration of the system, represented by the coordinates $q_1^0, q_2^0, \ldots, q_n^0$. Now, if the system is in equilibrium with zero initial velocities $\dot{q}_k$, the system will remain in equilibrium indefinitely.

Study of small oscillations

A particle is in *equilibrium if the net force F acting on it is zero. If it is at rest at such a* point, it stays there forever because it has zero acceleration and so can never begin to move. The equilibrium may be *stable or unstable. At a stable equilibrium point, when* the particle is displaced slightly, the direction of the force is such as to tend to push it back toward the equilibrium point. That is, for small displacements, the force is always *opposite in direction to the displacement.*

At an unstable equilibrium point, a particle displaced slightly from equilibrium experiences a force directed away from the equilibrium position (i.e., the *same direction* as the displacement), and the particle tends to move farther and farther from the equilibrium point.

The conditions for stable and unstable equilibrium can be expressed simply in terms of the force and potential energy functions *F(x) and V(x). Suppose* x_o *is an equilibrium* point, so that $F(x_o) = 0$. *We expand F(x) in a Taylor series about this point:*

$$F(x) = F(x_o) + \left[\frac{dF}{dx}\right]_{x=x_o}(x - x_o) + \frac{1}{2!}\left[\frac{d^2F}{dx^2}\right]_{x=x_o}(x - x_o)^2 + \cdots .$$

Study of small oscillations

The first term is zero because x_o *is an equilibrium point. If x - x_o is very small, then* the second term is the dominant one. Thus for points very near the equilibrium position,

$$F(x) \cong \left[\frac{dF}{dx}\right]_{x=x_o}(x - x_o) \quad .$$

We see that if *dF/dx is positive at xo, F is positive when x is greater than x_o and* negative when *x is less than x_o. In each case the force tends to push the particle farther* from the equilibrium position. We conclude that when *dF/dx is positive at x_o , this is a* point of *unstable equilibrium. A similar argument for the opposite case shows that when dF/dx is negative at x_o, it is a point of stable equilibrium.*

These conditions can also be expressed in terms of the potential-energy function *V(x).* Expanding it in a Taylor series about x_o, *we get*

$$V(x) = V(x_o) \quad + \quad \left[\frac{dV}{dx}\right]_{x=x_o}(x - x_o) \quad + \frac{1}{2!}\left[\frac{d^2V}{dx^2}\right]_{x=x_o}(x - x_o)^2 \quad + \cdots \; .$$

By selecting the origin at the coordinate system at the equilibrium position, we obtain x_o=0 and V(x_o)=V(0)=0
Using $(-dV/dx)_{x = xo}$ and neglecting higher order terms we get $V(x) = \frac{1}{2}kx^2$
Where *k*>0 is the constant value of d^2V/dx^2 at the equilibrium point $x = x_o$*:*

Study of small oscillations

The kinetic energy of the system is given by

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}^2$$

So, for one dimensional case, the Lagrangian *L* of the system, slightly displaced from equilibrium, is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The equation of motion, therefore is of the form

$$m\ddot{x} + kx = 0$$

Which is the equation of motion of a simple harmonic oscillator having the general solution

$$x = a\cos(\omega t - \delta)$$

where $\omega = \sqrt{(k/m)}$ is the frequency of oscillation of the system.

Thus, the frequency is of oscillation is simply the square root of the ratio of the coefficient of x^2 in the expression of potential energy and that of v^2 in the expression of the kinetic energy.

Study of small oscillations

In terms of the coordinate x_o and the initial speed v_0 the expression for the amplitude a and the phase δ is given by

$$a=\sqrt{(x_0^2 + v^2/\omega^2)}$$

and $$\delta=\tan^{-1}(-v_0/\omega x_0).$$

The solution may also be written in terms of complex quantities as

$$x=\mathrm{Re}[Ae^{\,i\omega t}], \qquad \text{where } A=ae^{\,i\,\delta}$$

The complex amplitude containing the information of the phase.m

The total energy of the system is given by

$$E=(1/2)m\,\omega^2 a^2.$$

Thus on being displaced from the equilibrium position, a system in stable equilibrium executes, for small displacements and amplitude, harmonic oscillations about the equilibrium position.

Systems performing small oscillations

Example: Two masses m_1 and m_2 are joined by a spring of force constant k. The spring gets compressed and released so that the system vibrates with a frequency ω. Find the value of ω.

Solution. The Lagrangian of the vibrating system is

$$L = \frac{1}{2}\mu v^2 - \frac{1}{2}kx^2 = \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2 \quad (1)$$

where μ is the reduced mass of the system and x is the extension (or the compression) of the spring.

$$\therefore \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Lagrange's equation for the vibrating system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\text{or,} \quad \mu\ddot{x} + kx = 0, \quad \text{using (1)}$$

$$\text{or,} \quad \ddot{x} + (k/\mu)x = 0$$

$$\text{or,} \quad \ddot{x} + \omega^2 x = 0, \quad \text{where } \omega = \sqrt{k/\mu}$$

The above equation is that of an S.H.M with frequency ω given by

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

Suggested readings:

1. *Classical Mechanics*, H. Goldstein, C.P. Poole, J.L. Safko, 3rd Edn. 2002, Pearson Education
2. *Mechanics*, L. D. Landau and E. M. Lifshitz, 1976, Pergamon
3. Classical Mechanics, J C Upadhyay, 2014, Himalaya Publishing House Pvt,. Ltd.