Economics Honours (Semester II)

Mathematical Economics -I

Maxima and Minima – Constrained Optimization

Meaning of Constrained Optimization

The act of maximizing or minimizing a given objective function subject to the fulfilment of certain conditions (also called constraints) is called constrained optimization. The basic format for a constrained optimization problem can be stated as follows:

a. **maximise $f(x)$ subject to $g(x) \geq 0$, and**
b. **minimize $f(x)$ subject to $g(x) \leq 0$**

- where $f(\mathbf{x})$ is the objective function in $\mathbf{x}$ and $g(\mathbf{x})$ is the stated constraint defined in the same variables.
- In a maximization problem, the constraint $g(\mathbf{x})$ is said to satisfy the constraint qualification if there is at least one $\mathbf{x}$ such that $g(\mathbf{x}) > 0$.
- In a minimization problem the constraint $g(\mathbf{x})$ is said to satisfy the constraint qualification if there is at least one $\mathbf{x}$ such that $g(\mathbf{x}) < 0$.
- Under constrained optimization the values of the real variable is subject to the restrictions imposed by the constraint.
- Solution to the constrained optimization problem is obtained by determining the first order condition (necessary condition) and the second order conditions (sufficient conditions). The first order condition can be obtained by the Lagrange multiplier method and the differential method. The second order condition is obtained by the differential method.

The First Order Condition: Lagrange multiplier method

The Lagrangian multiplier is a kind of shadow price of the constraints. With the help of a Lagrangian multiplier, we at first convert the constrained problem into an unconstrained optimization problem. This new function is called the Langrangian function Suppose we have a maximization problem and the basic problem is stated as follows:

maximise $f(x)$

subject to $g(x) \geq 0$.

The Lagrangian function, $L(x, \lambda)$, associated with the basic problem is defined as:

$$L(x, \lambda) = f(x) + \lambda\, g(x)$$

where λ is the Lagrangian multiplier.

Procedure to construct the Lagrangian function and solve the function

1) Identify the decision variables, x, and the objective function, f(x)
2) Write the constraint in the form $g(x) \geq 0$ in case of a maximization problem and $g(x) \leq 0$ in case of a minimization problem. Also check that at least one x satisfies the strict inequality.
3) Construct the Lagrangian function: $L(x, \lambda) = f(x) + \lambda\, g(x)$
4) Differentiate $L(x, \lambda)$ with respect to each decision variable and set these partials equal to zero.
5) Set $g(x) = 0$ and combine this equation with the equations obtained from (4) to solve x^* and λ^*.
6) Check λ^* is non-negative. If the λ^*calculated in (5) is negative then we reject the equation $g(x^*) = 0$ and set $\lambda^* = 0$. Then solve the equations given by (4).

Example: Suppose we have

The objective function $\quad z = f(x,y)$

subject to $\quad g(x,y) = 0$

The Lagrangian function is given by $L = f(x,y) + \lambda\, g(x,y)$

Differentiating L with respect to x, y, λ respectively, the first-order conditions are given as follows:

$$\delta L/\delta x = f_x + \lambda g_x = 0 \quad \ldots\ldots\ldots\ldots(1)$$

$$\delta L/\delta y = f_y + \lambda g_y = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$\delta L/\delta \lambda = g(x,y) = 0 \quad \ldots\ldots\ldots\ldots\ldots(3)$$

[Note: $f_x = \delta f/\delta x$, $g_x = \delta g/\delta x$ $f_y = \delta f/\delta y$, $g_y = \delta g/\delta y$]

The First Order Condition: Differential method

In the above example the Lagrangian function is given as $L = f(x,y) + \lambda\, g(x,y)$.

Hence by the differential method we can state

$$dL = d(f + \lambda g)$$

$$= f_x dx + f_y dy + \lambda g_x dx + \lambda g_y dy = 0$$

$$(f_x + \lambda g_x)dx + (f_y + \lambda g_y)\, dy = 0$$

If x and y are independent variables, then the necessary condition for the equation to be zero are that the coefficients of dx and dy be equal to zero. So we have

$$f_x + \lambda g_x = 0$$

$$f_y + \lambda g_y = 0$$

Here we find the results are the same conditions as obtained by the Lagrange multiplier method.

The Second –order conditions: Differential method

From the first-order differential method we had seen

$$dL = d(f + \lambda g)$$

$$\text{i.e. } dL = df + \lambda dg$$

Hence the second-order derivative is written as:

$$d^2L = d(df + \lambda dg)$$

In most economic problems it is found that $g(x,y) = 0$ is linear. Then $d(dg) = 0$

Hence d^2L becomes

$$d^2L = d(df + \lambda dg)$$

$$= d^2f + 0$$

$$= d^2f$$

$$\text{Thus } d^2L = (f_x dx + f_y dy)^2$$

$$= f_{xx}\, dx^2 + f_{yy} dy^2 + 2\, f_{xy}\, dxdy$$

If $d^2L < 0$, it maximizes the objective function subject to the constraint and

If $d^2L > 0$, it minimizes the objective function subject to the constraint.

Problem1. Starting with the utility function U= U(X,Y) and the budget constraint $P_xX + P_YY = M$, derive the first order equilibrium condition.

The problem can be stated as

maximize $U = U(X,Y)$

subject to $P_xX + P_YY = M$

The Lagrangian function can be written as

$L = U(X,Y) + \lambda\, (M - P_xX - P_YY)$

Partially Differentiating L with respect to X

$\delta L/\delta X = \delta U/\delta X - \lambda P_x = 0$ …………(1)

Partially Differentiating L with respect to Y

$\delta L/\delta Y = \delta U/\delta Y - \lambda P_Y = 0$ ………….(2)

Partially Differentiating L with respect to λ

$\delta L/\delta \lambda = M - P_xX - P_YY = 0$ …………….(3)

Dividing equation (1) by equation (2), we obtain

$$\frac{\delta U/\delta X}{\delta U/\delta Y} = \frac{\lambda P_X}{\lambda P_Y}$$

or,

$$\frac{MU_x}{MU_Y} = \frac{P_x}{P_Y}$$

or,

$MRS_{XY} = P_{x\,/}\,P_Y$

which is the necessary condition for equilibrium to exist.

Given the constraint function is a linear function; the second order derivative of the constraint function is zero. Hence for the sufficient condition,

we have, $d^2L = d^2U < 0,$

or, $U_{xx}\,dx^2 + U_{yy}dy^2 + 2\,U_{xy}\,dxdy < 0$

Problem 2

Given a production function $q = f(x_1, x_2)$ and a cost constraint $C = r_1\,x_1 + r_2\,x_2$, find the first and second order conditions for maximum output. Here x_i are the inputs, r_i are the input prices and C is the total cost.

The Lagrangian function is $L = f + \lambda\,(C - r_1\,x_1 - r_2\,x_2)$

Differentiating with respect to x_1 and x_2, the first order condition is given as follows:

$\delta L/\delta\, x_1 = f_1 - \lambda\, r_1 = 0$

$\delta L/\delta\, x_2 = f_2 - \lambda\, r_2 = 0$

which may be shown as $f_1/r_1 = f_2/r_2$

Given the constraint function is a linear function; the second order derivative of the constraint function is zero. Hence for the second order derivative of the maximization problem is given as follows:

we have, $d^2L = d^2q < 0,$

or, $f_{11}\,dx_1^2 + f_{22}dx_2^{\ 2} + 2\,f_{12}\,dx_1\,dx_2 < 0$

*N.B. Explanations of all the theories and examples have been taken from the following references:

References:

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