4 UNIT FOUR: Transportation and Assignment problems

4.1 Objectives

By the end of this unit you will be able to:

- formulate special linear programming problems using the transportation model.
- define a balanced transportation problem
- develop an initial solution of a transportation problem using the Northwest Corner Rule
- use the Stepping Stone method to find an optimal solution of a transportation problem
- formulate special linear programming problems using the assignment model
- solve assignment problems with the Hungarian method.

4.2 Introduction

In this unit we extend the theory of linear programming to two special linear programming problems, the **Transportation** and **Assignment Problems**. Both of these problems can be solved by the simplex algorithm, but the process would result in very large simplex tableaux and numerous simplex iterations.

Because of the special characteristics of each problem, however, alternative solution methods requiring significantly less mathematical manipulation have been developed.

4.3 The Transportation problem

The general transportation problem is concerned with determining an optimal strategy for distributing a commodity from a group of supply centres,such as factories, called *sources*, to various receiving centers, such as warehouses, called *destinations*, in such a way as to minimise total distribution costs.

Each source is able to supply a fixed number of units of the product, usually called the *capacity* or *availability*, and each destination has a fixed demand, often called the *requirement.*

Transportation models can also be used when a firm is trying to decide where to locate a new facility. Good financial decisions concerning facility location also attempt to minimize total transportation and production costs for the entire system.

4.3.1 Setting up a Transportation problem

To illustrate how to set up a transportation problem we consider the following example;

Example 4.1
A concrete company transports concrete from three plants, 1, 2 and 3, to three construction sites, A, B and C.

The plants are able to supply the following numbers of tons per week:

Plant	*Supply (capacity)*
1	*300*
2	*300*
3	*100*

The requirements of the sites, in number of tons per week, are:

Construction site	*Demand (requirement)*
A	*200*
B	*200*
C	*300*

The cost of transporting 1 ton of concrete from each plant to each site is shown in the figure 8 in Emalangeni per ton.

For computational purposes it is convenient to put all the above information into a table, as in the simplex method. In this table each row represents a source *and each column represents a* destination.

Sites

	To / *From*	*A*	*B*	*C*	*Supply (Availability)*
Plants	***1***	*4*	*3*	*8*	*300*
	2	*7*	*5*	*9*	*300*
	3	*4*	*5*	*5*	*100*
	Demand (requirement)	*200*	*200*	*300*	

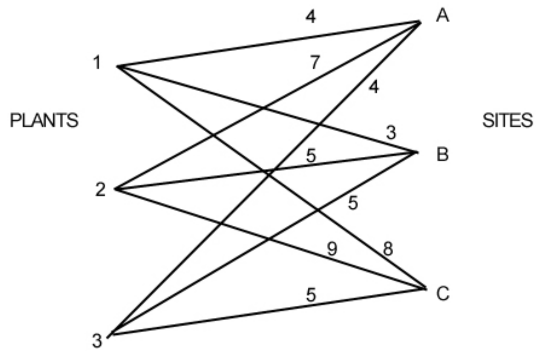


Figure 8: Constructing a transportation problem

4.3.2 Mathematical model of a transportation problem

Before we discuss the solution of transportation problems we will introduce the notation used to describe the transportation problem and show that it can be formulated as a linear programming problem.

We use the following notation;

$x_{ij} =$	the number of units to be distributed from source i to destination j $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$;
$s_i =$	supply from source i;
$d_j =$	demand at destination j;
$c_{ij} =$	cost per unit distributed from source i to destination j

With respect to Example 4.1 the decision variables x_{ij} are the numbers of tons transported from plant i (where $i = 1, 2, 3$) to each site j (where j = A, B, C)

A basic assumption is that the distribution costs of units from source i to destination j is directly proportional to the number of units distributed. A typical **cost and requirements table** has the form shown on Table 4.

Let Z be total distribution costs from all the m sources to the n destinations. In example 4.1 each term in the objective function Z represents the total cost of tonnage transported on one route. For example, in the route $2 \longrightarrow C$, the term in $9x_{2C}$, that is:

$$(\text{Cost per ton} = 9) \quad \times \quad (\text{number of tons transported} = x_{2C})$$

		Destination				
		1	2	$\dots$	n	*Supply*
	1	c_{11}	c_{12}	$\dots$	c_{1n}	s_1
	2	c_{21}	c_{22}	$\dots$	c_{2n}	s_2
Source	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
	m	c_{m1}	c_{m2}	$\dots$	c_{mn}	s_m
Demand		d_1	d_2	$\dots$	d_n	

Table 4: *Cost and requirements table*

Hence the objective function is:

$$\begin{aligned} Z &= 4x_{1A} + 3x_{1B} + 8x_{1C} \\ &+ 7x_{2A} + 5x_{2B} + 9x_{2C} \\ &+ 4x_{3A} + 5x_{3B} + 5x_{3C} \end{aligned}$$

Notice that in this problem the total supply is 300 + 300 + 200 = 700 and the total demand is 200 + 200 + 300 = 700. Thus

Total supply = total demand.

In mathematical form this expressed as

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \tag{47}$$

This is called a **balanced problem** . In this unit our discussion shall be restricted to the balanced problems.

In a balanced problem all the products that can be supplied are used to meet the demand. There are no slacks and so all constraints are *equalities* rather than *inequalities* as was the case in the previous unit.

The formulation of this problem as a linear programming problem is presented as

$$\text{Minimise} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}, \tag{48}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad \text{for} \quad i = 1, 2, \dots, m \tag{49}$$

$$\sum_{i=1}^{n} x_{ij} = d_j, \qquad \text{for} \quad j = 1, 2, \dots, n \tag{50}$$

and

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j.$$

Any linear programming problem that fits this special formulation is of the transportation type, regardless of its physical context. For many applications, the supply and demand quantities in the model will have integer values and implementation will require that the distribution quantities also be integers. Fortunately, the unit coefficients of the unknown variables in the constraints guarantee an optimal solution with only integer values.

4.3.3 Initial solution - Northwest Corner Rule

The initial basic feasible solution can be obtained by using one of several methods. We will consider only the **North West corner rule** of developing an initial solution. Other methods can be found in standard texts on linear programming.

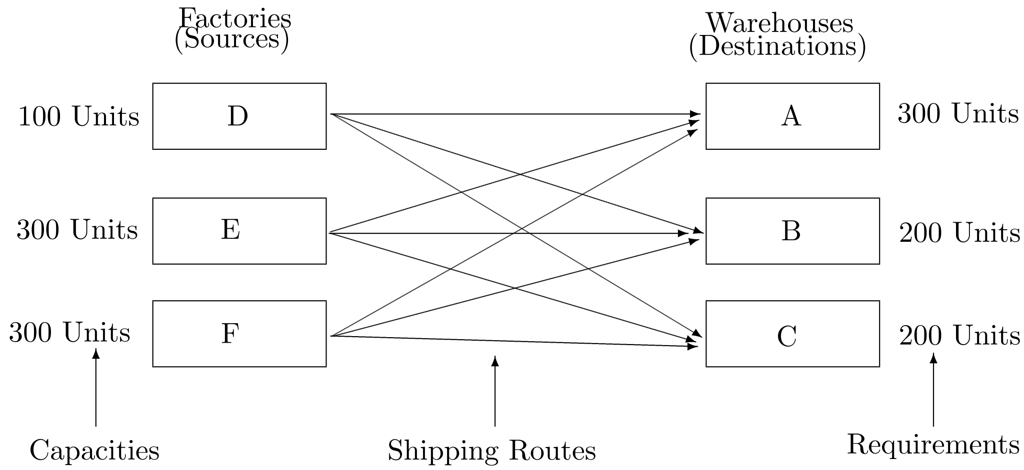
The procedure for constructing an initial basic feasible solution selects the basic variables one at a time. The North West corner rule begins with an allocation at the top left-hand corner of the tableau and proceeds systematically along either a row or a column and make allocations to subsequent cells until the bottom right-hand corner is reached, by which time enough allocations will have been made to constitute an initial solution.

The procedure for constructing an initial solution using the North West Corner rule is as follows:

NORTH WEST CORNER RULE

1. Start by selecting the cell in the most “North-West” corner of the table.
2. Assign the maximum amount to this cell that is allowable based on the requirements and the capacity constraints.
3. Exhaust the capacity from each row before moving down to another row.
4. Exhaust the requirement from each column before moving right to another column.
5. Check to make sure that the capacity and requirements are met.

Let us begin with an example dealing with Executive Furniture corporation, which manufactures office desks at three locations: D, E and F. The firm distributes the desks through regional warehouses located in A, B and C (see the Network format diagram below)



It is assumed that the production costs per desk are identical at each factory. The only relevant costs are those of shipping from each source to each destination. The costs are shown in Table 5

From \ To	A	B	C
D	\$5	\$4	\$3
E	\$8	\$4	\$3
F	\$9	\$7	\$5

Table 5: Transportation Costs per desk for Executive Furniture Corp.

We proceed to construct a transportation table and label its various components as show in Table 6.

We can now use the Northwest corner rule to find an initial feasible solution to the problem. We start in the upper left hand cell and allocate units to shipping routes as follows:

From \ To	A	B	C	Capacity
D	5	4	3	**100**
E	8	4	3	**300**
F	9	7	5	**300**
Requirements	**300**	**200**	**200**	**700**

Table 6: Transportation Table for Executive Furniture Corporation

1. Exhaust the supply (factory capacity) of each row before moving down to the next row.
2. Exhaust the demand (warehouse) requirements of each column before moving to the next column to the right.
3. Check that all supply and demand requirements are met.

The initial shipping assignments are given in Table 7

From \ To	A	B	C	Factory Capacity
D	100			100
E	200	100		300
F		100	200	300
Warehouse Requirements	300	200	200	700

Table 7: Initial Solution of the North West corner Rule

This initial solution can also be presented together with the costs per unit as shown in the Table 8.

We can compute the cost of this shipping assignment as follows;

Therefore, the initial feasible solution for this problem is $4200.

Example 4.2
Consider a transportation problem in which the cost, supply and demand values are presented in Table 10.

(a) Is this a balanced problem? Why?

To / From	A	B	C	Capacity
D	5 100	4	3	**100**
E	8 200	4 100	3	**300**
F	9	7 100	5 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 8: Representing the initial feasible solution with costs

ROUTE		UNITS		PER UNIT		TOTAL
FROM	TO	SHIPPED	×	COST ($)	=	COST ($)
D	A	100		5		500
E	A	200		8		1600
E	B	100		4		400
F	B	100		7		700
F	C	200		5		1000
						Total 4200

Table 9: Calculation of costs of initial shipping assignments

(b) *Obtain the initial feasible solution using the North-West Corner rule.*

Solution:

(a) *We calculate the total supply and total demand.*
Total supply = 14 + 10 + 15 + 13 = 52
Total demand = 10 + 15 + 12 + 15 = 52
Since the total supply is equal to the total demand we conclude that the problem is balanced.

(b) *The allocations according to the North-West corner rule are shown in Table 11 The initial feasible solution is*

$$\textit{Total Cost} = 10 \cdot 10 + 4 \cdot 30 + 10 \cdot 15 + 1 \cdot 30 + 12 \cdot 20 + 2 \cdot 20 + 13 \cdot 45 = \$1265$$

Note that this is not necessarily equal to the optimal solution.

		Destination				
		1	2	3	4	Supply
	1	10	30	25	15	14
Source	2	20	15	20	10	10
	3	10	30	20	20	15
	4	30	40	35	45	13
	Demand	10	15	12	15	

Table 10: Supply and Demand values for Transportation problem

	1	2	3	4	Supply
1	10	4			14
2		10			10
3		1	12	2	15
4				13	13
Demand	10	15	12	15	

Table 11: Initial feasible solution

4.4 Exercises 4.1: Northwest Corner rule

In each of the following problems check whether the solution is balanced or not then use the North West Corner rule to find the basic feasible solution.

1.

TO / FROM	1	2	3	Supply
1	3	2	0	45
2	1	5	0	60
3	5	4	0	35
Demand	50	60	30	

2.

TO / FROM	1	2	3	Supply
1	5	4	3	100
2	8	4	3	300
3	9	7	5	300
Demand	300	200	200	

3.

FROM \ TO	1	2	3	4	Supply
A	12	13	4	6	500
B	6	4	10	11	700
C	10	9	12	4	800
Demand	400	900	200	500	

4.

FROM \ TO	1	2	3	4	Supply
1	10	30	25	15	14
2	20	15	20	10	10
3	10	30	20	20	15
4	30	40	35	45	13
Demand	10	15	12	15	

4.4.1 Optimality test - the Stepping Stone method

The next step is to determine whether the current allocation at any stage of the solution process is optimal. We will present one of the methods used to determine optimality of and improve a current solution. The method derives its name from the analogy of crossing a pond using stepping stones. The occupied cells are analogous to the stepping stones, which are used in making certain movements in this method.

The five steps of the **Stepping-Stone Method** are as follows:

STEPPING-STONE METHOD

1. Select an unused square to be evaluated.
2. Beginning at this square, trace a closed path back to the original square via squares that are currently being used (only horizontal or vertical moves allowed). You can only change directions at occupied cells!.
3. Beginning with a plus (+) sign at the unused square, place alternative minus (-) signs and plus signs on each corner square of the closed path just traced.
4. Calculate an **improvement index**, I_{ij} by adding together the unit cost figures found in each square containing a plus sign and then subtracting the unit costs in each square containing a minus sign.
5. Repeat steps 1 to 4 until an improvement index has been calculated for all unused squares.
   - If all indices computed are greater than or equal to zero, an optimal solution has been reached.
   - If not, it is possible to improve the current solution and decrease total shipping costs.

4.4.2 The optimality criterion

If all the cost index values obtained for all the currently unoccupied cells are nonnegative, then the current solution is optimal. If there are negative values the solution has to be improved. This means that an allocation is made to one of the empty cells (unused routes) and the necessary adjustments in the supply and demand effected accordingly.

To see how the Stepping-Stone method works we apply these steps to the Furniture Corporation example to evaluate the shipping routes.

Steps 1-3 Beginning with the D-B route, we first trace a closed path using only currently occupied squared (see Table 12) and then place alternate plus signs and minus signs in the corners of this path.

Step 4 An improvement index I_{ij} for the D-B route in now computed by adding unit costs in squares with plus signs and subtracting costs in squares with minus signs. Thus

$$I_{DB} = +4 - 5 + 8 - 4 = +3$$

This means that for every desk shipped via the D-B route, total transportation costs will increase by \$3 over their current level.

From \ To	A	B	C	Capacity
D	5 100 - ←	Start 4 +	3	**100**
E	↓ 8 200 + →	↑ 4 - 100	3	**300**
F	9	7 100	5 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 12: Evaluating the D-B route

Step 5 Next we consider the D-C unused route. The closed path we use is (see Table 13)

$$+DC - DA + EA - EB + FB - FC$$

The D-C improvement index is

$$I_{DC} = +3 - 5 + 8 - 4 + 7 - 5 = +4$$

From \ To	A	B	C	Capacity
D	5 100 - ←	4 ⟵ ⟵ ←	Start 3 +	**100**
E	↓ 8 200 + →	4 - 100	↑ 3 ↑	**300**
F	9	↓ 7 + ⟶ → 100	↑ 5 - 200	**300**
Requirements	**300**	**200**	**200**	**700**

Table 13: Evaluating the D-C route

The other two routes may be evaluated in a similar fashion

E-C route: closed path = +EC - EB + FB - FC

$$I_{EC} = +3 - 4 + 7 - 5 = +1$$

FA route: closed path = +FA - FB + EB - EA

$$I_{FA} = +9 - 7 + 4 - 8 = -2$$

Because the I_{FA} index is negative, a cost saving may be attained by making use of the FA route i.e the FA cell can be improved. The Stepping-Stone path used to evaluate the route FA is shown in Table 14

From \ To	A	B	C	Capacity
D	100 [5]	[4]	[3]	**100**
E	200 - ← [8]	+ 100 [4]	[3]	**300**
F	Start ↓ + → [9]	↑ - 100 [7]	200 [5]	**300**
Requirements	**300**	**200**	**200**	**700**

Table 14: Stepping-Stone Path used to evaluate FA route

The next step, then is to ship the maximum allowable number of units on the new route (FA route). What is the maximum quantity that can be shipped on the money-saving route? The quantity is found by referring to the closed path of plus signs and minus signs drawn for the route and selecting the *smallest number* found in those squares containing minus signs. To obtain a new solution, that number is added to all squares on the closed path with plus signs and subtracted from all squares on the path assigned minus signs. All other squares are left unchanged. The new solution is shown in Table 15.

From \ To	A	B	C	Capacity
D	100 [5]	[4]	[3]	**100**
E	100 [8]	200 [4]	[3]	**300**
F	100 [9]	[7]	200 [5]	**300**
Requirements	**300**	**200**	**200**	**700**

Table 15: Improved solution: Second solution

The shipping cost for this new solution is

$$100 \cdot 5 + 100 \cdot 8 + 200 \cdot 4 + 100 \cdot 9 + 200 \cdot 4 = \$4000$$

This solution may or may not be optimal. To determine whether further improvement is possible, we return to the first five steps to test each square that is now unused. The four improvement indices - each representing an available shipping route are as follows:

$$\begin{aligned} \text{D to B} &= I_{DB} = 4 - 5 + 8 - 4 = +\$3 \\ & \quad (\text{Closed path} : +DB - DA + EA - EB) \end{aligned}$$

$$
\begin{aligned}
\text{D to C} &= I_{DC} = 3 - 5 + 9 - 5 = +\$2 \\
&\quad (\text{Closed path} : +DC - DA + FA - FC) \\
\text{E to C} &= I_{EC} = 3 - 8 + 9 - 5 = -\$1 \\
&\quad (\text{Closed path} : +EC - EA + FA - FC) \\
\text{F to B} &= I_{FB} = 7 - 4 + 8 - 9 = +\$2 \\
&\quad (\text{Closed path} : +FB - EB + EA - FA)
\end{aligned}
$$

Hence, an improvement can be made by shipping the maximum allowable number of units from E to C (see Table 16).

From \ To	A	B	C	Capacity
D	100 [5]	[4]	[3]	**100**
E	- ↓ 100 [8]	← ← 200 [4]	← + Start [3]	**300**
F	↓ 100 + → [9]	→ → → [7]	↑ → - 200 [5]	**300**
Requirements	**300**	**200**	**200**	**700**

Table 16: Path to evaluate the E-C route

The improved solution is shown in Table 17. The total cost for the third solution is

$$100 \cdot 5 + 200 \cdot 4 + 100 \cdot 3 + 200 \cdot 9 + 100 \cdot 5 = \$3900$$

To determine if the current solution is optimal we calculate the improvement indices - each

From \ To	A	B	C	Capacity
D	100 [5]	[4]	[3]	**100**
E	[8]	200 [4]	100 [3]	**300**
F	200 [9]	[7]	100 [5]	**300**
Requirements	**300**	**200**	**200**	**700**

Table 17: Improved solution: Third solution

representing an available shipping route - as follows:

$$\text{D to B} = I_{DB} = 4 - 5 + 9 - 5 + 3 - 4 = +\$2$$

$$\begin{aligned}
& \text{(Closed path: } + DB - DA + FA - FC + EC - EB) \\
\text{D to C} \quad = \quad & I_{DC} = 3 - 5 + 9 - 5 = +\$2 \\
& \text{(Closed path: } + DC - DA + FA - FC) \\
\text{E to A} \quad = \quad & I_{EA} = 8 - 9 + 5 - 3 = +\$1 \\
& \text{(Closed path: } + EA - FA + FC - EC) \\
\text{F to B} \quad = \quad & I_{FB} = 7 - 5 + 3 - 4 = +\$1 \\
& \text{(Closed path: } + FB - FC + EC - EB)
\end{aligned}$$

Table 17 contains the optimal solution because each improvement index for the Table is greater than or equal to zero.

4.5 Summary

In this section we discussed the formulation of transportation problems and their methods of solution. We used the North West corner rule to obtain the initial feasible solution and the Stepping-Stone method to find the optimal solution. We restricted focus to balanced transportation problems where it is assumed that the total supply is equal to total demand.

4.6 Exercises 4.2: Transportation problems

1. A company has factories at A, B and C which supply warehouses at D, E and F. Weekly factory capacities are 200, 160 and 90 units respectively. Weekly warehouse requirements (demands) are 180, 120 and 150 units respectively. Unit shipping costs (in Emalangeni) are as follows:

Factory	D	E	F	Capacity
A	16	20	12	**200**
B	14	8	18	**160**
C	26	24	16	**90**
Demand	**180**	**120**	**150**	**450**

   Determine the optimum distribution for this company to minimize shipping costs. [E5920]

2. A Timber company ships pine flooring to three building supply houses from its mills in Bhunya, Mondi and Pigg's Peak. Determine the best transportation schedule for the data given below using the Northwest corner rule and the Stepping Stone method. [E230]

TO / FROM	*Supply House 1*	*Supply House 2*	*Supply House 3*	*Mill Capacity (tons)*
Bhunya	3	3	2	**25**
Mondi	4	2	3	**40**
Pigg's Peak	3	2	3	**30**
Supply House Demand (tons)	**30**	**30**	**35**	**95**

3. A trucking company has a contract to move 115 truckloads of sand per week between three sand-washing plants W,X and Y, and three destinations, A,B and C. Cost and volume information is given below. Compute the optimal transportation cost.

To / From	Project A	Project B	Project C	Supply
Plant W	5	10	10	35
Plant X	20	30	20	40
Plant Y	5	8	12	40
Demand	45	50	20	

[C=1345]

4. In each of the following cases write down the North West corner solution and use the Stepping Stone method to find the minimal cost.

(a)

TO / FROM	D	E	F	Capacity
A	8	6	9	**20**
B	6	3	8	**30**
C	10	7	9	**70**
Demand	**90**	**20**	**10**	**120**

[E970]

(b)

TO / FROM	D	E	F	Capacity
A	2	2	3	**4**
B	2	1	6	**6**
C	1	3	4	**8**
Demand	**2**	**5**	**11**	**18**

[E48]

(c)

FROM \ TO	D	E	F	Capacity
A	5	2	2	**7**
B	7	3	4	**5**
C	6	4	3	**3**
Demand	**4**	**5**	**6**	

4.7 Assignment Problem

The **assignment problem** refers to the class of linear programming problems that involve determining the most efficient assignment of

- people to projects
- salespeople to territories
- contracts to bidders
- jobs to machines, etc.

The objective is most often to minimize total costs or total time of performing the tasks at hand.

One important characteristic of assignment problems is that only one job or worker is assigned to one machine or project. An example is the problem of a taxi company with 4 taxis and 4 passengers. Which taxi should collect which passenger in order to minimize costs?

Each assignment problem has associated with it a table, or matrix. Generally, the rows contain the objects or people we wish to assign, and the columns comprise the tasks or things we want them assigned to. The numbers in the table are the costs associated with each particular assignment.

An assignment problem can be viewed as a transportation problem in which

- the capacity from each source (or person to be assigned) is 1 and
- the demand at each destination (or job to be done) is 1.

As an illustration of the assignment problem, let us consider the case of a Fix-It-Shop, which has just received three new rush projects to repair: (1) a radio, (2) a toaster oven, and (3) a broken coffee table. Three repair persons, each with different talents and abilities, are available to do the jobs. The owner of the shop estimates what it will cost in wages to assign each of the workers to each of the three projects. The costs which are shown in Table 18 differ because the owner believes that each worker will differ in speed and skill on these quite varied jobs.

Table 19 summarizes all six assignment options. The table also shows that the least-cost solution would be to assign Cooper to project 1, Brown to project 2, and Adams to project 3, at a total cost of \$25.

The owner's objective is to assign the three projects to the workers in a way that will result in the lowest cost to the shop. Note that the assignment of people to projects must be on a one-to-one basis; each project will be assigned exclusively to one worker only.

PERSON	PROJECT 1	2	3
Adams	$11	$14	$6
Brown	8	10	11
Cooper	9	12	7

Table 18: Repair costs of the Fix-It-Shop assignment problem

PROJECT ASSIGNMENT				
1	2	3	LABOUR COSTS ($)	TOTAL COSTS ($)
Adams	Brown	Cooper	11 + 10 +7	28
Adams	Cooper	Brown	11 + 12 +11	34
Brown	Adams	Cooper	8 + 14 + 7	29
Brown	Cooper	Adams	8 + 12 + 6	26
Cooper	Adams	Brown	9 + 14 + 11	34
Cooper	Brown	Adams	9 + 10 + 6	25

Table 19: Assignment alternatives and Costs of Fix-It-Shop assignment problem

Special algorithms exist to solve assignment problems. The most common is probably the **Hungarian** solution method. The Hungarian method of assignment provides us with an efficient means of finding the optimal solution without having to make a direct comparison of every assignment option. It operates on a principle of matrix reduction, which means that by subtracting and adding appropriate numbers in the cost table or matrix, we can reduce the problem to a matrix of *opportunity costs*. Opportunity costs show the relative penalties associated with assigning any person to a project as opposed to making the best or least-cost assignment. We would like to make assignments such that the opportunity cost for each assignment is zero.

The steps involved in the Hungarian method are outlined below.

THE HUNGARIAN METHOD

1. *Find the opportunity cost table by*
   (a) Subtracting the smallest number in each row of the original cost table or matrix from every number in that row.
   (b) Then subtracting the smallest number in each column of the table obtained in part (a) from every number in that column.
2. *Test the table resulting from step 1 to see whether an optimal assignment can be made.* The procedure is to draw the minimum number of vertical and horizontal straight lines necessary to cover all zeros in the table. If the number of lines equals either the number of rows or columns, an optimal assignment can be made. If the number of lines is less than the number of rows or columns, we proceed to step 3.
3. *Revise the present opportunity cost table.* This is done by subtracting the smallest number not covered by a line from every other uncovered number. This same smallest number is also added to any number(s) lying at the intersection of the horizontal and vertical lines. We then return to step 2 and continue the cycle until an optimal assignment is possible.

Let us now apply the three steps to the Fix-It-Shop assignment example.

The original cost table for the problem is given in Table 20

	PROJECT		
PERSON	1	2	3
Adams	11	14	6
Brown	8	10	11
Cooper	9	12	7

Table 20: Initial Table

	PROJECT		
PERSON	1	2	3
Adams	5	8	0
Brown	0	2	3
Cooper	2	5	0

Table 21: Row reduction (part a)

After the row reduction (Step 1 part a) we get the cost Table 21.

Taking the costs in Table 21 and subtracting the the smallest number in each column from each number in that column results in the total opportunity costs given in Table 22. This step is the column reduction of Step 1 part (b)

If we draw vertical and horizontal straight lines (Step 2) to cover all the zeros in Table 22 we get Table 23. Since the number of lines is less than the number of rows or columns an optimal assignment cannot be made.

Since Table 23 doesn't give an optimal solution we revise the table. This is accomplished by subtracting the smallest number not covered by a line from all numbers not covered by

PERSON	PROJECT 1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Table 22: Column Reduction (Step 1 part b)

PERSON	PROJECT 1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Table 23: Testing for an optimal solution

a straight line. This same smallest number is then added to every number (including zeros) *lying in the intersection* on any two lines. The smallest uncovered number in Table 23 is 2, so this value is subtracted from each of the four uncovered numbers. A 2 is also added to the number that is covered by the intersecting horizontal and vertical lines. The results of this step are shown in Table 24

To test now for an optimal assignment, we return to Step 2 and find the minimum number of lines necessary to cover all zeros in the revised opportunity cost table. Because it requires three lines to cover the zeros (see Table 25), an optimal assignment can be made.

PERSON	PROJECT 1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

Table 24: Revised opportunity cost table

PERSON	PROJECT 1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

Table 25: Optimality test on the revised table

Finally, we make the allocation. Note that only one assignment will be made from each row or column. We use this fact to proceed to making the final allocation as follows:

(a) Find a row or column with only one zero cell.

(b) Make the assignment corresponding to that zero cell.

(c) Eliminate that row and column from the table.

(d) Continue until all the assignments have been made.

For our Fix-It-Shop problem these steps are summarized in Table 26.

	FIRST ASSIGNMENT				SECOND ASSIGNMENT				THIRD ASSIGNMENT		
	1	2	3		1	2	3		1	2	3
Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$
Brown	0	0	5	Brown	0	0	5	Brown	0	$\boxed{0}$	5
Cooper	0	1	0	Cooper	$\boxed{0}$	1	0	Cooper	$\boxed{0}$	1	0

Table 26: Making the final assignment

To interpret the table we recall that our objective was to minimize costs, there is only one assignment that Adams can go to where the opportunity costs are \$0. That is to assign Adams Project 3. If Adams gets assigned to Project 3, then there is only one project left where the opportunity cost is \$0 for Cooper. Therefore Cooper gets assigned to Project 1. This leaves Brown being assigned to Project 2, where the opportunity costs are \$0.

The optimal allocation is to assign Adams to Project 3, Brown to Project 2, and Cooper to Project 1. The total labour cost of this assignment are computed from the original cost table (see Table 18). They are as follows:

ASSIGNMENT	COST (\$)
Adams to Project 3	6
Brown to Project 2	10
Cooper to Project 1	9
Total cost	25

Example 4.3 *Suppose we have to allocate 4 tasks (1,2,3,4) between 4 people (W,X,Y,Z). The costs are set out in the following table:*

	Task			
Person	*1*	*2*	*3*	*4*
W	*8*	*20*	*15*	*17*
X	*15*	*16*	*12*	*10*
Y	*22*	*19*	*16*	*30*
Z	*25*	*15*	*12*	*9*

The entries in the table denote the costs of assigning a task to a particular person.

Solution: Step 1 of the Hungarian method involves the following parts:

(a) subtract the minimum value from each column (see Table 27)

(b) subtract the minimum value from each column (see Table 28)

	Task			
Person	1	2	3	4
W	0	12	7	9
X	5	6	2	0
Y	6	3	0	14
Z	16	6	3	0

Table 27: Subtract the minimum value from each row

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	2	0
Y	6	0	0	14
Z	16	3	3	0

Table 28: subtract the minimum value from each column

The next step is to check whether optimal assignment can be made. This is done by finding the minimum *number of lines necessary to cross-out all the zero cells in the table. If this is equal to n (the number of people/tasks) then the solution has been found. The minimum number of lines necessary to cross through all the zeros (see Table 29)is 3 ¡ n = 4 so that an optimal allocation has not been found.*

*(*Note that there may be more than one way to draw the lines through the zero cells. It does not matter which way you choose as long as there is no alternative way involving fewer lines*)*

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	$\boxed{2}$	0
Y	6	0	0	14
Z	16	3	3	0

Table 29: Checking if an optimal assignment can been made

Next we revise the table by

(a) Finding the minimum uncovered cell. Table 29 shows that the minimum uncovered cell has a value of 2

(b) Subtracting the value obtained in (a) (i.e subtract 2) from all the uncovered cells.

(c) Adding to all the cells at the intersection of the two lines.

The result of the above steps is given in Table 30.

We then check if the revised allocation is optimal. This is done by finding the minimum number of lines required to cover all zeros (see Table 31).

This time the minimum number of lines necessary to cross through all the zeros is n = 4 so that an optimal allocation has been found.

To make the final allocation we use the following steps.

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

Table 30: Revising the Table

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

Table 31: Checking for optimality

- *Find a row or column with only one zero cell.*
- *Make the assignment corresponding to that zero cell.*
- *eliminate that row and column from the table.*
- *Continue until all assignments have been found.*

	Task			
Person	*1*	*2*	*3*	*4*
W	*0*	*7*	*5*	*9*
X	*5*	*1*	*0*	*0*
Y	*8*	*0*	*0*	*16*
Z	*16*	*1*	*1*	*0*

- *Assign person W to task 1 and eliminate row W and column 1.*
- *Assign person Y to task 2 and eliminate row Y and column 2.*
- *Assign person Z to task 4 and eliminate row Z and column 4.*
- *This leaves final person X assigned to remaining task 3.*

From the original cost table, we can determine the costs associated with the optimal assignment:

$$Total \quad Cost = 48$$

4.8 Maximization Assignment Problems

Some assignment problems are phrased in terms of **maximizing** the payoff, profit, or effectiveness of an assignment instead of *minimization* costs. It is easy to obtain an equivalent minimization problem by converting all numbers in the table to opportunity costs; efficiencies to inefficiencies,etc. This is achieved through *subtracting every number in the original payoff table from the largest single number in the number.* The transformed entries represent opportunity costs; it turns out that minimizing the opportunity costs produces the same assignment as the original maximization problem. Once the optimal assignment for this transformed problem has been computed, the total payoff or profit is found by adding the original payoffs of those cells that are in the original assignment.

Example. The British Navy wishes to assign four ships to patrol four sectors of the North Sea. In some areas ships are to be on the outlook for illegal fishing boats, and in other sectors to watch for enemy submarines, so the commander rates each ship in terms of its profitable efficiency in each sector. These relative efficiencies are illustrated in Tables 32. On the basis of the ratings shown, the commander wants to determine the patrol assignments producing the greatest overall efficiencies.

	SECTOR			
SHIP	A	B	C	D
1	20	60	50	55
2	60	30	80	75
3	80	100	90	80
4	65	80	75	70

Table 32: Efficiencies of British Ships in Patrol sectors

	SECTOR			
SHIP	A	B	C	D
1	80	40	50	45
2	40	70	20	25
3	20	0	10	20
4	35	20	25	30

Table 33: Opportunity Costs of British Ships

We start by converting the maximizing efficiency table into a minimization opportunity cost table. This is done by subtracting each rating from 100, the largest rating in the whole table. The resulting opportunity costs are given in Table 33.

Next, we follow steps 1 and 2 of the assignment algorithm. The smallest number is subtracted from every number in that row to give Table 34; and then the smallest number in each column is subtracted from every number in that column as shown in Table 35.

	SECTOR			
SHIP	A	B	C	D
1	40	0	10	5
2	20	50	0	5
3	20	0	10	20
4	15	0	5	10

Table 34: Row opportunity costs for the British Navy Problem

	SECTOR			
SHIP	A	B	C	D
1	25	0	10	0
2	5	50	0	0
3	5	0	10	15
4	0	0	5	5

Table 35: Total opportunity costs for the British Navy Problem

The minimum number of straight lines needed to cover all zeros in this total opportunity cost table is four. Hence an optimal assignment can be made. The optimal assignment is ship 1 to sector D, ship 2 to sector C, ship 3 to sector B, and ship 4 to sector A.

The overall efficiency, computed from the original efficiency data Table 32, can now be shown:

ASSIGNMENT	EFFICIENCY
Ship 1 to Sector D	55
Ship 2 to Sector C	80
Ship 3 to Sector B	100
Ship 4 to Sector A	65
Total Efficiency	300

4.9 Summary

In this section we discussed the Hungarian method for solving both maximization and minimization assignment problems.

4.10 Exercises 4.3: Minimization Assignment Problems

1. Three accountants, Phindile, Rachel and Sibongile, are to be assigned to three projects, 1, 2 and 3. The assignment costs in units of E1000 are given in the table below.

		Project		
		1	**2**	**3**
	P	15	9	12
Accountant	**R**	7	5	10
	S	13	4	6

2. Joy Taxi has four taxis, 1,2,3 and 4, and there are four customers, P, Q, R and S requiring taxis. The distance between the taxis and the customers are given in the table below, in Kilometres. The Taxi company wishes to assign the taxis to customers so that the distance traveled is a minimum.

		Customers			
		P	Q	R	S
	1	10	8	4	6
Taxis	**2**	6	4	12	8
	3	14	10	8	2
	4	4	14	10	8

3. Four precision components are to be shaped using four machine tools, one tool being assigned to each component. The machining times, in minutes, are given in the table below.

		Component			
		1	**2**	**3**	**4**
	A	21	20	39	36
Machine Tool	**B**	25	22	24	25
	C	36	22	36	26
	D	34	21	25	39

4. In a job shop operation, four jobs may be performed on any of four machines. The hours required for each job on each machine are presented in the following table. The plant supervisor would like to assign jobs so that total time in minimized. Use the assignment method to find the best solution.

	MACHINE			
JOB	W	X	Y	Z
A12	10	14	16	13
A15	12	13	15	12
B2	9	12	12	11
B9	14	16	18	16

Answer: A12 to W, A15 to Z, B2 to Y, B9 to Z, 50 hours.

4.11 Exercises 4.4: Maximization Assignment Problems

1. A head of department has four lecturers to assign to pure maths (1), mechanics (2), statistics (3) and Quantitative techniques (4). All of the teachers have taught the courses in the past and have been evaluated with a score from 0 to 100. The scores are shown in the table below.

	1	**2**	**3**	**4**
Peters	80	55	45	45
Radebe	58	35	70	50
Tsabedze	70	50	80	65
Williams	90	70	40	80

The head of department wishes to know the optimal assignment of teachers to courses that will maximize the overall total score. Use the Hungarian algorithm to solve this problem. [$P \to 1$, $R \to 3$, $T \to 4$, $W \to 2$ Max Score = 285]

2. A department store has leased a new store and wishes to decide how to place four departments in four locations so as to maximize total profits. The table below gives the profits, in thousands of emalangeni, when the departments are allocated to the various locations. Find the assignment that maximizes total profits.

		Location			
		1	**2**	**3**	**4**
Department	Shoes	20	16	22	18
	Toys	25	28	15	21
	Auto	27	20	23	26
	Housewares	24	22	23	22

3. The head of the business department, has decided to apply the Hungarian method in assigning lecturers to courses next semester. As a criterion for judging who should teach each course, the head of department reviews the past two years' teaching evaluations. All the four lecturers have taught each of the courses at one time or another during the two year period. The ratings are shown in the table below.
   Find the best assignment of lecturers to courses to maximize the overall teaching rating. Total Rating = 335

	COURSE			
LECTURER	STATISTICS	MANAGEMENT	FINANCE	ECONOMICS
Dlamini	90	65	95	40
Khumalo	70	60	80	75
Masuku	85	40	80	60
Nxumalo	55	80	65	55