Lecture 2 : Convergence of a Sequence, Monotone sequences

In less formal terms, a sequence is a set with an order in the sense that there is a first element, second element and so on. In other words for each positive integer 1,2,3, $\ldots$, we associate an element in this set. In the sequel, we will consider only sequences of real numbers.

Let us give the formal definition of a sequence.

Definition : A function $f : \{1, 2, 3, \ldots\} \to \mathbb{R}$ is called a sequence of real numbers. We write $f(n) = x_n$, then the sequence is denoted by $x_1, x_2, \ldots$, or simply by (x_n). We call x_n the nth term of the sequence or the value of the sequence at n.

Some examples of sequences:

1. $(n) = 1, 2, 3, \ldots$
2. $(\frac{1}{n}) = 1, \frac{1}{2}, \frac{1}{3}, \ldots$
3. $(\frac{(-1)^n}{n}) = -1, \frac{1}{2}, \frac{-1}{3}, \ldots$
4. $(1 - \frac{1}{n}) = 0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots$
5. $(1 + \frac{1}{10^n}) = 1.1, 1.01, 1.001, \ldots$
6. $((-1)^n) = -1, +1, -1, +1, \ldots$

Before giving the formal definition of convergence of a sequence, let us take a look at the behaviour of the sequences in the above examples.

The elements of the sequences $(\frac{1}{n})$, $(1-1/n)$ and $(1+1/10^n)$ seem to "approach" a single point as n increases. In these sequences the values are either increasing or decreasing as n increases, but they "eventually approach" a single point. Though the elements of the sequence $((-1)^n/n)$ oscillate, they "eventually approach" the single point 0. The common feature of these sequences is that the terms of each sequence "accumulate" at only one point. On the other hand, values of the sequence (n) become larger and larger and do not accumulate anywhere. The elements of the sequence $((-1)^n)$ oscillate between two different points -1 and 1; i.e., the elements of the sequence come close to -1 and 1 "frequently" as n increases.

Convergence of a Sequence

Let us distinguish sequences whose elements approach a single point as n increases (in this case we say that they converge) from those sequences whose elements do not. Geometrically, it is clear that if the elements of the sequence (x_n) come eventually inside *every* ϵ-neighbourhood $(x_0 - \epsilon, x_0 + \epsilon)$ of x_0 then (x_n) approaches x_0.

Let us now state the formal definition of convergence.

Definition : We say that a sequence (x_n) converges if there exists $x_0 \in I\!R$ such that for every $\epsilon > 0$, there exists a positive integer N (depending on ϵ) such that

$$x_n \in (x_0 - \epsilon, x_0 + \epsilon) \ \ \text{(or } |x_n - x_0| < \epsilon\text{)} \text{ for all } n \geq N.$$

It can be easily verified that if such a number x_0 exists then it is unique. In this case, we say that the sequence (x_n) converges to x_0 and we call x_0 the limit of the sequence (x_n). If x_0 is the limit of (x_n), we write $\lim_{n\to\infty} x_n = x_0$ or $x_n \to x_0$.

Examples : 1. Let us show that the sequence $(\frac{1}{n})$ in Example 1 has limit equal to 0. For arbitrary $\epsilon > 0$, the inequality

$$|x_n| = \frac{1}{n} < \epsilon$$

is true for all $n > \frac{1}{\epsilon}$ and hence for all $n > N$, where N is any natural number such that $N > \frac{1}{\epsilon}$. Thus for any $\epsilon > 0$, there is a natural number N such that $|x_n| < \epsilon$ for every $n \geq N$.

2. The sequence in Example 4 converges to 1, because in this case

$$|1 - x_n| = |1 - \frac{n-1}{n}| = \frac{1}{n} \leq \epsilon$$

for all $n > N$ where N is any natural number greater than $\frac{1}{\epsilon}$.

Remark: The convergence of each sequence given in the above examples is verified directly from the definition. In general, verifying the convergence directly from the definition is a difficult task. We will see some methods to find limits of certain sequences and some sufficient conditions for the convergence of a sequence.

The following three results enable us to evaluate the limits of many sequences.

Limit Theorems

Theorem 2.1: *Suppose $x_n \to x$ and $y_n \to y$. Then*

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$
3. $\frac{x_n}{y_n} \to \frac{x}{y}$ if $y \neq 0$ and $y_n \neq 0$ for all n.

Example : Let $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}$. Then $x_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \to 1$

Theorem 2.2 : (*Sandwich Theorem*) *Suppose that $(x_n), (y_n)$ and (z_n) are sequences such that $x_n \leq y_n \leq z_n$ for all n and that $x_n \to x_0$ and $z_n \to x_0$. Then $y_n \to x_0$.*

Proof: Let $\epsilon > 0$ be given. Since $x_n \to x_0$ and $z_n \to x_0$, there exist N_1 and N_2 such that

$$x_n \in (x_0 - \epsilon, x_0 + \epsilon) \ \text{ for all } \ n \geq N_1$$

and

$$z_n \in (x_0 - \epsilon, x_0 + \epsilon) \ \text{ for all } \ n \geq N_2.$$

Choose $N = max\{N_1, N_2\}$. Then, since $x_n \leq y_n \leq z_n$, we have

$$y_n \in (x_0 - \epsilon, x_0 + \epsilon) \ \text{ for all } \ n \geq N.$$

This proves that $y_n \to x_0$. □

Examples : 1. Since $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$, by sandwich theorem $\frac{\sin n}{n} \to 0$.

2. Let $y_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$. Then $\frac{n.n^2}{n^3+2n} \leq y_n \leq \frac{n.n^2}{n^3+n+1}$ and hence $y_n \to 1$.

3. Let $x \in \mathbb{R}$ and $0 < x < 1$. We show that $x^n \to 0$. Write $x = \frac{1}{1+a}$ for some $a > 0$. Then by Bernoulli's inequality, $0 < x^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$. By sandwich theorem $x^n \to 0$.

4. Let $x \in \mathbb{R}$ and $x > 0$. We show that $x^{\frac{1}{n}} \to 1$. Suppose $x > 1$ and $x^{\frac{1}{n}} = 1 + d_n$ for some $d_n > 0$. By Bernoulli's inequality, $x = (1 + d_n)^n > 1 + nd_n > nd_n$ which implies that $0 < d_n < \frac{x}{n}$ for all $n \in \mathbb{N}$. By sandwich theorem $d_n \to 0$ and hence $x^{\frac{1}{n}} \to 1$. If $0 < x < 1$, let $y = \frac{1}{x}$ so that $y^{\frac{1}{n}} \to 1$ and hence $x^{\frac{1}{n}} \to 1$.

5. We show that $n^{\frac{1}{n}} \to 1$. Let $n^{\frac{1}{n}} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n > 1$ for $n > 1$. By Binomial theorem, if $n > 1$, $n \geq 1 + \frac{1}{2}n(n-1)k_n^2$. Therefore $n - 1 \geq \frac{1}{2}n(n-1)k_n^2$ and hence $k_n^2 \leq \frac{2}{n}$. By sandwich theorem $k_n \to 0$ and therefore $n^{\frac{1}{n}} \to 1$.

The following result, called ratio test for sequences, can be applied to certain type of sequences for convergence.

Theorem 2.3: *Let (x_n) be a sequence of real numbers such that $x_n > 0$ for all n and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$. Then*

1. *if $\lambda < 1$ then $\lim_{n\to\infty} x_n = 0$,*
2. *if $\lambda > 1$ then $\lim_{n\to\infty} x_n = \infty$.*

Proof : 1. Since $\lambda < 1$, we can find an r such that $\lambda < r < 1$. As $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$, there exists n_0 such that $\frac{x_{n+1}}{x_n} < r$ for all $n \geq n_0$. Hence,

$$0 < x_{n+n_0} < rx_{n+n_0-1} < r^2x_{n+n_0-2} < \cdots < r^nx_{n_0}.$$

Note that $\lim_{n\to\infty} r^n = 0$ as $0 < r < 1$. So by the sandwich theorem $x_n \to 0$.

2. Since $\lambda > 1$, we can find $r \in \mathbb{R}$, such that $1 < r < \lambda$. Arguing along the same lines as in 1., we get $n_0 \in \mathbb{N}$, such that $x_{n+1} > rx_n$ for all $n \geq n_0$. Similarly, $x_{n+n_0} > r^nx_{n_0}$. Since $r > 1$, $\lim_{n\to\infty} r^n = \infty$ and therefore $\lim_{n\to\infty} x_n = \infty$. $\square$

Examples : 1. Let $x_n = \frac{n}{2^n}$ and $y_n = \frac{2^n}{n!}$. Then $x_n \to 0$ as $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{2}$. We do similarly for y_n.

2. Let $x_n = ny^{n-1}$ for some $y \in (0, 1)$. Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = y$, $x_n \to 0$.

3. Let $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$. Repeat the argument as in the previuos problem and show that $x_n \to 0$.

4. Let $b > 1$ and $x_n = \frac{b^n}{n^2}$. Then $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = b$. Therefore, $\lim_{n\to\infty} x_n = \infty$.

5. In the previous theorem if $\lambda = 1$ then we cannot make any conclusion. For example, consider the sequences $(n), (\frac{1}{n})$ and $(2 + \frac{1}{n})$.

In the previous results we could guess the limit of a sequence by comparing the given sequence with some other sequences whose limits are known and then we could verify that our guess is correct. We now give a simple criterion for the convergence of a sequence (without having any knowledge of its limit).

Before presenting a criterion (a sufficient condition), let us see a necessary condition for the convergence of a sequence.

Theorem 2.4: *Every convergent sequence is a bounded sequence, that is the set* $\{x_n : n \in \mathbb{N}\}$ *is bounded.*

Proof : Suppose a sequence (x_n) converges to x. Then, for $\epsilon = 1$, there exist N such that

$$|x_n - x| \leq 1 \quad \text{for all} \;\; n \geq N.$$

This implies $|x_n| \leq |x| + 1$ for all $n \geq N$. If we let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|\},$$

then $|x_n| \leq M + |x| + 1$ for all n. Hence (x_n) is a bounded sequence. □

Remark : The condition given in the previous result is necessary but not sufficient. For example, the sequence $((-1)^n)$ is a bounded sequence but it does not converge.

One naturally asks the following question:

Question : Boundedness $+$ (??) $\Rightarrow$ **Convergence.**

We now find a condition on a bounded sequence which ensures the convergence of the sequence.

Monotone Sequences

Definition : We say that a sequence (x_n) is increasing if $x_n \leq x_{n+1}$ for all n and strictly increasing if $x_n < x_{n+1}$ for all n. Similarly, we define decreasing and strictly decreasing sequences. Sequences which are either increasing or decreasing are called monotone.

The following result is an application of the least upper bound property of the real number system.

Theorem 2.5: *Suppose* (x_n) *is a bounded and increasing sequence. Then the least upper bound of the set* $\{x_n : n \in \mathbb{N}\}$ *is the limit of* (x_n).

Proof: Suppose $\sup\limits_n x_n = M$. Then for given $\epsilon > 0$, there exists n_0 such that $M - \epsilon \leq x_{n_0}$. Since (x_n) is increasing, we have $x_{n_0} \leq x_n$ for all $n \geq n_0$. This implies that

$$M - \epsilon \leq x_n \leq M \leq M + \epsilon \;\; \text{for all } n \geq n_0.$$

That is $x_n \to M$. □

For decreasing sequences we have the following result and its proof is similar.

Theorem 2.6: *Suppose* (x_n) *is a bounded and decreasing sequence. Then the greatest lower bound of the set* $\{x_n : n \in \mathbb{N}\}$ *is the limit of* (x_n).

Examples: 1. Let $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$ for $n > 1$. Then use induction to see that $0 \leq x_n \leq 2$ and (x_n) is increasing. Therefore, by previous result (x_n) converges. Suppose $x_n \to \lambda$. Then $\lambda = \sqrt{2 + \lambda}$. This implies that $\lambda = 2$.

2. Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$. Note that $\frac{x_{n+1}}{x_n} < 1$. Hence the sequence is decreasing. Since $x_n > 0$, the sequence is bounded below. Therefore (x_n) converges. Suppose $x_n \to \lambda$. Then $\lambda = \frac{\lambda}{2} + 2$. Therefore, $\lambda = 2$.