Chapter 7

# Sequences of Functions

Sequences of functions play in important role approximation theory. They can be used to show a solution of a differential equation exists. We recall in Chapter Three we define a sequence to be a function whose domain is the natural numbers. Thus, if $f_n(x) : \mathcal{D} \to \mathbb{R}$ for each $n \in \mathbb{N}$, then $\{f_n\}_{n\in\mathbb{N}}$ is sequence of functions. We need a notion of convergence.

**Definition 7.1.** We say the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined on a set $\mathcal{D}$ converges *pointwise*, if and only if for each $x \in \mathcal{D}$, the sequence of real numbers $\{f_n(x)\}_{n\in\mathbb{N}}$ converges. We set $f(x) = \lim_{n\to\infty} f_n(x)$.

We would like to know what $f$ inherits from the $f_n$. In general, the answer is not much.

**Example 7.2.** Consider $f_n(x) = x^n$ on $[0, 1]$ and $n \in \mathbb{N}$. Note that $f_n \in C^\infty(0, 1)$ for all $n \in \mathbb{N}$. However, the limit

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

is not even continuous on $[0, 1]$.

**Example 7.3.** Consider $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$ and $n \in \mathbb{N}$. Once again $f_n \in C^\infty(\mathbb{R})$ for all $n \in \mathbb{N}$ and the pointwise limit is $f(x) = 0$ is also infinitely differentiable. However, $f_n'(x) = \sqrt{n}\cos nx$ has no limit and certainly does not converge to $f'(x) = 0$. Thus, we see

$$(\lim_{n\to\infty} f_n(x))' \neq \lim_{n\to\infty} f_n'(x).$$

**Example 7.4.** Back in Chapter One in Example 1.5 we considered the sequence of functions given by $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$. One checks the limit is $f(x) = 0$. Once again both the sequence and the limit have infinitely many derivatives. However,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} -\frac{1}{2}e^{-nx^2}\Big|_0^1 = \frac{1}{2},$$

while

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 \lim_{n\to\infty} 0\, dx = 0.$$

Thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

So what does $f$ inherit from $f_n$? We need a stronger form of convergence.

**Definition 7.5.** A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on $\mathcal{D}$ is uniformly convergent on $\mathcal{D}$ if and only if, for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathcal{D}$ and $n \geq N$.

**Lemma 7.6.** *A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on $\mathcal{D}$ does not converge uniformly to its pointwise limit $f$ if there exists a $\epsilon_0 > 0$ and a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ and a sequence $\{x_k\}_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0$ for all $k \in \mathbb{N}$.*

**Proof.** In logic notation uniform convergence is

$$(\forall\epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in \mathcal{D})(\forall n \geq N)(|f(x) - f_n(x)| < \epsilon).$$

The negation of this is

$$(\exists\epsilon_0 > 0)(\forall N \in \mathbb{N})(\exists x \in \mathcal{D})(\exists n \geq N)(|f(x) - f_n(x)| \geq \epsilon_0).$$

That is

$$(\exists\epsilon_0 > 0)(\forall k \in \mathbb{N})(\exists x_k \in \mathcal{D})(\exists n_k \geq k)(|f(x_k) - f_{n_k}(x_k)| \geq \epsilon_0)$$

as required. □

**Example 7.7.** Show that $f_n(x) = x^n$ on $[0, 1]$ and $n \in \mathbb{N}$ does not converge uniformly on $[0, 1]$.

**Proof.** We construct the sequence $\{x_k\}$ to converge to the *trouble* spot. Here, the trouble is at $x = 1$ (a graph of several $f_n$ will help to see this). We pick $\epsilon_0 = 1/2$, $n_k = k$ and $x_k = (1/2)^{1/k}$. Then

$$|f(x_k) - f_{n_k}(x_k)| = |f_{n_k}(x_k)| = \frac{1}{2} \geq \epsilon_0,$$

and $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $[0, 1]$. □

**Example 7.8.** Show that $f_n(x) = x^n$ on $[0, 1/2]$ and $n \in \mathbb{N}$ does converge uniformly on $[0, 1/2]$.

**Proof.** We just need to show that the $N$ in the definition of convergence does not depend on $x$. The pointwise limit $f = 0$. We calculate

$$|f_n(x) - f(x)| = |f_n(x) - 0| \leq \left(\frac{1}{2}\right)^n.$$

Thus, given any $\epsilon > 0$ we choose $N \in \mathbb{N}$ so that $(1/2)^n < \epsilon$ for all $n \geq N$ (how do we do this?). Then $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1/2]$ and $n \geq N$. □

**Example 7.9.** Show that the sequence of functions given by $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$ and $n \in \mathbb{N}$ does not converge uniformly on $[0, 1]$.

**Proof.** A graph of the $f_n$ is given in Example 1.5. The peak of the curves (the maximum of $f_n$ occurs at $x = 1/\sqrt{2n}$, and $f_n(1/\sqrt{2n}) = \sqrt{\frac{n}{2}}e^{-1/2}$. This suggest choosing $\epsilon_0$ equal to any positive number, $n_k = k$, $x_k = 1/\sqrt{2k}$. Then

$$|f_{n_k}(x_k) - f(x_k)| = |f_{n_k}(x_k)| = \sqrt{\frac{k}{2}}e^{-1/2}$$

which tends to infinity. □

We might expect uniform convergence to imply $f$ inherits the properties of $f_n$. This is not quite true.

**Example 7.10.** Consider again $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$ with $n \in \mathbb{N}$. The pointwise limit is $f(x) = 0$. Moreover, the convergence is uniform since

$$|f_n(x) - f(x)| = |f_n(x) - 0| \leq \frac{1}{\sqrt{n}}$$

which can be made small independently of $x$. However, recall that $f_n'$ does not converge to $f'$.

**Example 7.11.** The sequence of functions $f_n(x) = x/n$ on $\mathbb{R}$, $n \in \mathbb{N}$, does not converge uniformly on $\mathbb{R}$. Indeed, set $\epsilon_0 = 1$, $n_k = k$, and $x_k = k$. Then $|f_{n_k}(x_k) - f(x_k)| = |1 - 0| \geq \epsilon$. However, $f$ inherits the continuity of the $f_n$ and $f_n'$ converges to $f'$.

## 7.1. Preservation Theorems

In this section we find conditions on the sequence $\{f_n\}_{n\in\mathbb{N}}$ so that its properties are retained by the pointwise limit $f$.

**Theorem 7.12.** *Let $\mathcal{D}$ be any nonempty subset of $\mathbb{R}$ and suppose $f_n \in C(\mathcal{D})$ for each $n \in \mathbb{N}$. If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to $f$ on $\mathcal{D}$, then $f$ is continuous on $\mathcal{D}$.*

**Proof.** Since $f_n$ converges uniformly on $\mathcal{D}$, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon \tag{7.1}$$

for all $x \in \mathcal{D}$ and $n \geq N$. Since each $f_n$ is continuous on $\mathcal{D}$, for any $\epsilon > 0$, $x_0 \in \mathcal{D}$, and $n \in \mathbb{N}$, there exists $\delta(\epsilon, x_0, n)$ such that

$$|f_n(x) - f_n(x_0)| < \epsilon \tag{7.2}$$

for all $x \in \mathcal{D}$ and $|x - x_0| < \delta$.

We need to show $f$ is continuous on $\mathcal{D}$. Let $\epsilon > 0$ and $x_0 \in \mathcal{D}$. Equation (7.1) shows the existence of a $N_0 \in \mathbb{N}$ so that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $n \geq N_0$ and all $x \in \mathcal{D}$. Equation (7.2) provides a $\delta(\epsilon, x_0, N_0)$ such that

$$|f_{N_0}(x) - f_{N_0}(x_0)| < \frac{\epsilon}{3}$$

for all $x \in \mathcal{D}$, $|x - x_0| < \delta$. Thus

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_{N_0}(x) + f_{N_0}(x) - f_{N_0}(x_0) + f_{N_0}(x_0) - f(x_0)| \\ &\leq |f(x) - f_{N_0}(x)| + |f_{N_0}(x) - f_{N_0}(x_0)| + |f_{N_0}(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

for all $x \in \mathcal{D}$ with $|x - x_0| < \delta$, and hence $f \in C(\mathcal{D})$. ☐

**Example 7.13.** Show (again) that $f_n(x) = x^n$ on $[0,1]$, $n \in \mathbb{N}$, does not converge uniformly on $[0,1]$.

**Proof.** We note (again) that the $f_n$ are continuous. Since the pointwise limit $f$ is not continuous, the convergence cannot be uniform. ☐

**Example 7.14.** Consider

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

for $n \in \mathbb{N}$. The pointwise limit is $f = 0$, and it is easily seen that $f_n \to f$ uniformly on $[0,1]$. While the $f_n$ are discontinuous the limit $f$ is smooth. Thus uniform convergence preserves *good* properties, not bad, [**6**].

**Theorem 7.15.** *Suppose $f_n \in R[a,b]$ for all $n \in \mathbb{N}$. Furthermore, suppose $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to $f$ on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx. \tag{7.3}$$

**Proof.** We first show $f$ is integrable. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}$$

for all $n \geq N$ and $x \in [a,b]$. It follows that $U(P, f - f_n) < \epsilon/3$ and $L(P, f - f_n) > -\epsilon/3$ for any partition $P$ of $[a,b]$. Since $F_N$ is Riemann integrable, there exists a partition of $[a,b]$ so that

$$U(P, f_N) - L(P, f_N) < \epsilon/3.$$

For any two bounded functions $h, g$ and any partition, note that

$$U(P, h+g) \leq U(P,h) + U(P,g), \qquad L(P,h+g) \geq L(P,h) + L(P,g).$$

Thus

$$\begin{aligned} U(P,f) - L(P,f) &\leq U(P, f - f_N) + \Big(U(P, f_N) - L(P, f_N)\Big) - L(P, f - f_N) \\ &< 3\frac{\epsilon}{3}. \end{aligned}$$

Now that we know $f$ is Riemann integrable, a simple calculation shows

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq \int_a^b |f_n(x) - f(x)|\,dx < \int_a^b \frac{\epsilon}{3(b-a)} < \epsilon$$

for all $n \geq N$ and (7.3) follows. ☐

If we assume that $f$ is Riemann integrable, we may drop the uniform convergence.

**Theorem 7.16.** *(Bounded-Convergence Theorem) Let $f_n \in \mathcal{R}[a,b]$ for all $n \in \mathbb{N}$ and suppose the pointwise limit $f$ is also Riemann integrable on $[a,b]$. If $M$ exists so that $|f_n(x)| \leq M$ for all $x \in [a,b]$ and $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$$

**Proof.** For a proof at the level of this book see [**5**]. □

**Example 7.17.** Note that the bounded convergence theorem applies to the sequence of functions $f_n(x) = x^n$ on $[0,1]$ while Theorem 7.15 does not.

**Theorem 7.18.** *Suppose, for each $n \in \mathbb{N}$,*

i) $f_n \in C^1(I)$
ii) $\lim_{n\to\infty} f_n(x_0)$ *exists for some* $x_0 \in I$.
iii) $\{f_n'\}_{n\in\mathbb{N}}$ *converges uniformly on* $I$.

*Then*

$$\lim_{n\to\infty} f_n'(x) = \Big(\lim_{n\to\infty} f_n(x)\Big)'. \tag{7.4}$$

**Proof.** Suppose $f_n'$ converge uniformly on $I$ to $g$. By Theorem 7.12 $g$ is continuous on $I$. Moreover, by the Fundamental Theorem of Calculus

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(s)\,ds.$$

Theorem 7.15 applies to $f_n'$, and we therefore conclude $f_n(x)$ converges pointwise to, say, $f$. In fact, the previous equation shows

$$f(x) = f(x_0) + \int_{x_0}^x g(s)\,ds.$$

The Fundamental Theorem of Calculus applies again, and we see that $f' = g$. This is (7.4). □

## 7.2. Homework

**Exercise 7.1.** This is the same as Problem 6.27. Work it again using on of the theorems in this chapter. Suppose $f$ is continuous on $[0,1]$. Define $g_n(x) = f(x^n)$ for $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = f(0)$.

**Exercise 7.2.** Give an example of a sequence of continuous functions which converge to a continuous function but where the convergence is not uniform.

**Exercise 7.3.** Give an example of a sequence of functions discontinuous everywhere which converge uniformly to a continuous function.

**Exercise 7.4.** Let $f_n(x) = 1/(nx+1)$ and $g_n(x) = x/(nx+1)$ for $x \in (0,1)$. Show that $\{f_n\}_{n\in\mathbb{N}}$ does not converges uniformly on $(0,1)$, but $\{g_n\}_{n\in\mathbb{N}}$ does converge uniformly on $(0,1)$.

**Exercise 7.5.** Show that $f_n(x) = \sin^n(x)$ does not converge uniformly on $[0,\pi/2]$.

**Exercise 7.6.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on a nonempty set $E$ such that $f_n(x) \to f(x)$ uniformly on $E$ as $n \to \infty$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $E$ converging to an $x \in E$. Does it follow that $f_n(x_n)$ converges to $f(x)$?

**Exercise 7.7.** Consider the following functions defined for $x \geq 0$

(a) $\dfrac{x^n}{n}$,

(b) $\dfrac{x^n}{n+x^n}$,

(c) $\dfrac{x^n}{1+x^n}$,

(d) $\dfrac{x}{n}e^{-x/n}$.

Discuss the convergence and the uniform convergence of these sequences and the continuity of the limit functions. In the case on nonuniform convergence consider the sequences on an appropriate interval in $E$.

**Exercise 7.8.** Prove the following result. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on a set $\mathcal{D}$ and each $f_n$ is bounded, then $f_n$ is uniformly bounded. That is, there exists a $M$ such that $|f_n(x)| \leq M$ for all $x \in \mathcal{D}$ and $n \in \mathbb{N}$. Use this result to give another proof that the sequence in Example 7.9 does not converge uniformly.

**Exercise 7.9.** Prove that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on a set $\mathcal{D}$ if $\{M_n\}_{n\in\mathbb{N}}$ exists such that $M_n \to 0$ monotonically as $n \to \infty$ and, for all $x \in \mathcal{D}$, $|f_n(x) - f(x)| \leq M_n$ .

**Exercise 7.10.** Let $f_n(x) = x + 1/n$ on $\mathbb{R}$. Show that $f_n$ converges uniformly on $\mathbb{R}$, but $f_n^2$ does not.

**Exercise 7.11.** Suppose $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ converge uniformly on a set $\mathcal{D}$.

(a) Show that $\{f_n \pm g_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathcal{D}$.

(b) Show that in general $\{f_n\, g_n\}_{n\in\mathbb{N}}$ does not converge uniformly on $\mathcal{D}$.

(c) If each $f_n$ and $g_n$ are bounded, show that $\{f_n g_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathcal{D}$.

**Exercise 7.12.** If $0 < a < 2$, show that $\lim_{n\to\infty} \int_a^2 e^{-nx^2}\, dx = 0$. What happens if $a = 0$?

**Exercise 7.13.** Show that $f_n = \sqrt{x^2 + 1/n^2}$ converges uniformly to $|x|$ on $\mathbb{R}$. [Hint: Rationalize to show $|f_n(x) - \sqrt{x^2}| \leq 1/n$ for each $n$.

**Exercise 7.14.** If $f_n$ converges uniformly to $f$ on a set $E$ and each $f_n$ is continuous, prove $f$ is continuous. Show that the statement if false if either the uniform convergence or the continuity of the $f_n$ are dropped.

**Exercise 7.15.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$ and for $n \in \mathbb{N}$, let $f_n(x) = f(x + 1/n)$ for $x \in \mathbb{R}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to $f$ on $\mathbb{R}$.

**Exercise 7.16.** Prove the Monotone Convergence Theorem. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of monotone increasing Riemann integrable functions. That is, $f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \ldots$ for all $x \in [a, b]$. If the pointwise limit $f$ is Riemann integrable, then $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$.

**Exercise 7.17.** Prove Cauchy's Theorem for sequences of functions: a sequence of function $\{f_n\}_{n\in\mathbb{N}}$ on a domain $\mathcal{D}$ converges uniformly to $f$ on $\mathcal{D}$ if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy on $\mathcal{D}$. As in the case of sequences of real numbers, this provides a way to show a sequence of functions converges without having to find its pointwise limit.

**Exercise 7.18.** (Dini's Theorem) Suppose $f_n$ converges to $f$ pointwise on $[a, b]$ with both $f_n$ and $f$ continuous on $[a, b]$. If the sequence is monotone decreasing, $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in [a, b]$, then $f_n \to f$ uniformly on $[a, b]$.

**Exercise 7.19.** Let $g \in \mathcal{R}[a, b]$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence converging to $f$ uniformly on $[a, b]$. Prove $\lim_{n\to\infty} \int_a^b f_n(x)g(x)\, dx = \int_a^b f(x)g(x)\, dx$.

**Exercise 7.20.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions on a domain $\mathcal{D}$ converging uniformly on $\mathcal{D}$ to $f$. Show that $\lim_{n\to\infty} \sup(f_n(x)) = \sup(f(x))$.