Economics Honours (Semester II)

Mathematical Economics -I

Maxima and Minima – Unconstrained Optimization

***Increasing and Decreasing Functions**

As we know for any given function y = f (x), different values of the variable x gives rise to a possible set of values for the function f(x), defined as the domain of the function.

Example: Given the function y = f(x) =2x. Then for the set of values of x say

For $x \varepsilon \{1, 2, 3, 4, 5\}$,

$y \varepsilon \{2, 4, 6, 8, 10\}$

Thus $f(x) \varepsilon \{2, 4, 6, 8, 10\}$ is called the domain of the given function.

The first order derivative of f(x) i.e. f '(x) is used to determine whether the function is increasing or decreasing on any interval in its domain. If f '(x) > 0, then the function is said to be *increasing* at each point in the interval where the function exists. Again if f '(x) < 0, then the function is said to be *decreasing* at each point in the interval where the function exists.

Example: $y = f(x) = x^2$, hence first order derivative of y with respect to x is

dy/dx = f'(x) = 2x.

In this case at x = 2, f '(x) = 4.

However at x = -2, f '(x) = - 4.

This means at x =2, the function $y = f(x) = x^2$ is increasing and at x = -2, the same function is decreasing.

***Unconstrained Optimization of a Function**

In the function y=f(x), if for any given value of x, dy/dx = 0, then we determine an optimum point on the curve. This is also the *stationary* point on y for the given value of x.

Example: $y = f(x) = x^2$, hence first order derivative of y with respect to x is

dy/dx = f'(x) = 2x.

In this case at x = 0, f '(x) = 0. Hence the function in the example acquires an optimum value at x=0.

A function in general may have several optimum points or optima for different values of x. However the first order derivative of the function does not give us a clear idea if the value is maximum or minimum at a given point of x. For this purpose we need to understand the second order derivative of the function. Thus to determine the maxima or minima of a determined local optima of a function of the form y =f(x) at a value x = x*, we follow these necessary and sufficient conditions.

(a) A *necessary* condition for x* to be a *local optimum* of f(x) is that x* be a stationary point of f(x), i.e., dy/dx= f '(x) = 0

(b) A *sufficient* condition for a stationary point, x*, to be a local *maximum* is d^2y/dx^2, i.e. , f "(x) < 0 or to be a local *minimum* is or d^2y/dx^2 , i.e. f "(x) > 0.

This method of minimizing or maximizing an objective function that depends on real variables without any restrictions on the values of the variables is called **unconstrained optimization**. In an unconstrained optimization problem, the maxima and minima are found at stationary points.

Example 1:

Suppose $y = f(x) = x^2$, hence first order derivative of y with respect to x is

$dy/dx = f'(x) = 2x$.

In this case at $x = 0$, $f'(x) = 0$ and $f''(x) = 2$, i.e., $f''(x) > 0$

This means at x =0, the f (x) has a local optima with minimum value.

Example 2:

Suppose $y = f(x) = 4x - x^2$, hence first order derivative of y with respect to x is

$dy/dx = f'(x) = 4 - 2x$.

In this case at $x = 2$, $f'(x) = 0$ and $f''(x) = -2$, i.e., $f''(x) < 0$

This means at x =2, the f (x) has a local optima with maximum value.

In a function, the maximum and minimum are critical values which together are also known as the *extreme values* of the function.

*Local Optima and Global Optima

In a given function, y = f(x), if the domain of x is enlarged, more than one maximum value and more than one minimum value may occur. This can be explained with the help of Figure 1.

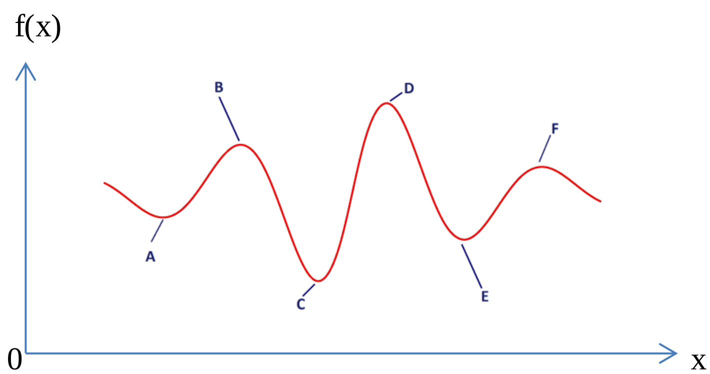

Figure 1: Local and Global Optima
Source: https://blog.fullstory.com/local-to-global-maxima/

In Fig 1, the points A, C, E exhibit minimum values of the function. All these points give us a local minimum for the function. A global minimum point on a function is defined as that local minimum on the function which gives the lowest value of all the local minima. Of all the local minimum values, point C denotes minimum of the minimum values. Hence C is both a local and global minimum.

Again in the same figure, the points B, D, F exhibit maximum values of the function. All these points give us a local maximum for the function. A global maximum point on a function is defined as that local maximum on the function which gives the highest value of all the local maxima. Of all the local maximum values, point D denotes maximum of the maximum values. Hence D is both a local and global maximum.

***Concavity, Convexity and Inflection**

Suppose we have a function in one variable. Given the function y = f(x), for a given value of x, the value of the second order derivative of the function, i.e. d^2y/dx^2 = f ”(x) has an additional implication. If f ”(x) > 0, then the function is not only called an increasing function but also concave upwards (also called convex downwards) in shape at the given value of x. If f ”(x) < 0, then the function is not only called decreasing function but also concave downwards (also called convex upwards) in shape at the given value of x. This is shown in Figure 2.

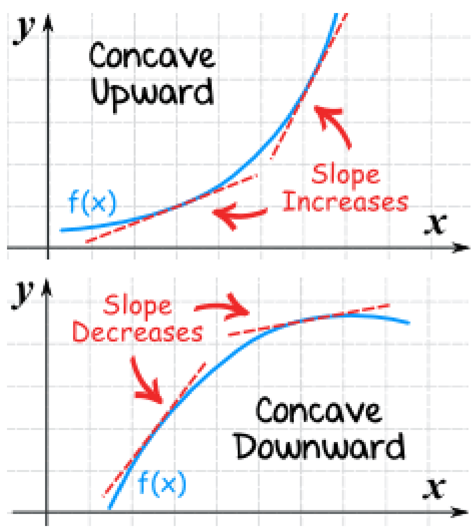

Figure 2. Concave and Convex functions in one variable
Source: http://en.statmania.info/2015/10/convex-concave.html

In the above figure, the curve which is concave upwards (convex downwards) in shape has f ”(x) > 0 at the points of tangency. Similarly the curve which is concave downwards (convex upwards) in shape has f ”(x) < 0 at the points of tangency.

*****_Important definitions_

- A function y = $f(x)$ is said to be concave if for every pair (x,y) and every α between zero and one we have $f[\alpha x + (1-\alpha)y] \geq \alpha f(x) + (1-\alpha)f(y)$
- A function y = $f(x)$ is said to be strictly concave if for every pair (x,y) and every α between zero and one we have $f[\alpha x + (1-\alpha)y] > \alpha f(x) + (1-\alpha)f(y)$
- A function y = $f(x)$ is said to be convex if for every pair (x,y) and every α between zero and one we have $f[\alpha x + (1-\alpha)y] \leq \alpha f(x) + (1-\alpha)f(y)$
- A function y = $f(x)$ is said to be strictly convex if for every pair (x,y) and every α between zero and one we have $f[\alpha x + (1-\alpha)y] < \alpha f(x) + (1-\alpha)f(y)$

*****_A special case involving concavity and convexity_

In the given function y =f(x), if at a particular value of x, say x = a, the curve changes its shape such that

at x =a , $f''(a) = 0$,

and as x crosses x =a, $f''(x)$ changes sign,

then the point on the curve corresponding to the given value of x =a is called the point of *inflection*. This is shown in figure 3,

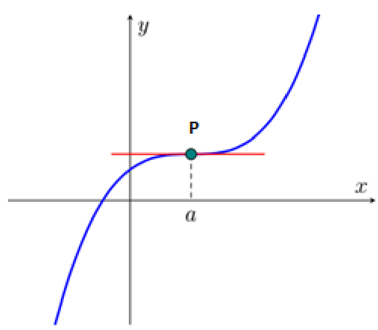

Figure 3: Point of inflexion.

Source: https://iitutor.com/inflection-points-points-of-inflection/

In the above figure,

at point P, for the value x = a, $f''(a) = 0$,

for points to the left of P on the curve i.e. for values x< a, the curve is concave downwards, hence $f''(a) < 0$,

for points to the right of P on the curve i.e. for values x> a, the curve is concave upwards, hence $f''(a) > 0$.

Thus point P is called the point of *inflection*.

Problems

1. Determine whether the curves are rising (increasing) or falling (decreasing) at the given points. Also find the x that gives y a stationary value

i. $y = 2x^2 - 6x + 2$. (a) x= 2, (b) x= 6

ii. $y = x^3 - 3x^2$ (a) x= 1, (b) x= 4

iii. $y = x^2 - 2x$ (a) x= -1, (b) x= 3

iv. $y = -x^2 + 4x$ (a) x= 1, (b) x= 5

2. Using the problems in 1 above, determine whether the curves are convex downward or convex upward.

3. Find the point(s) of inflection, if any, on the following curves.

a. $y = x^4 - 6x^2$

b. $y = 3x^3 - 6x^2$

c. $y = c\,e^{-x/2}$

d. $y = x^4 - 4x + 1$

4. Using the functions in Problems 1 and 3, find the extreme values, and show whether they are maximum or minimum.

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*N.B. Diagrams, Figures and Explanations of all the theories have been taken from the following references:

References

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