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Alokkumar De.

Theory of Errors: Systematic and Random Errors. Propagation of Errors. Normal Law of Errors. Standard and Probable Error.

Theory of Errors. **PART - I**

Error:

“The difference between an observed or calculated value and the true value.”

■ Webster.

One does not know the true value, usually.

Systematic error:

- “An error having a non-zero mean, so that its effect is not reduced when observations are averaged.”

■ Oxford

Systematic errors in experimental observations usually come from the measuring instruments [01].
Examples:

1. Offset or zero setting error
2. Multiplier or scale factor error

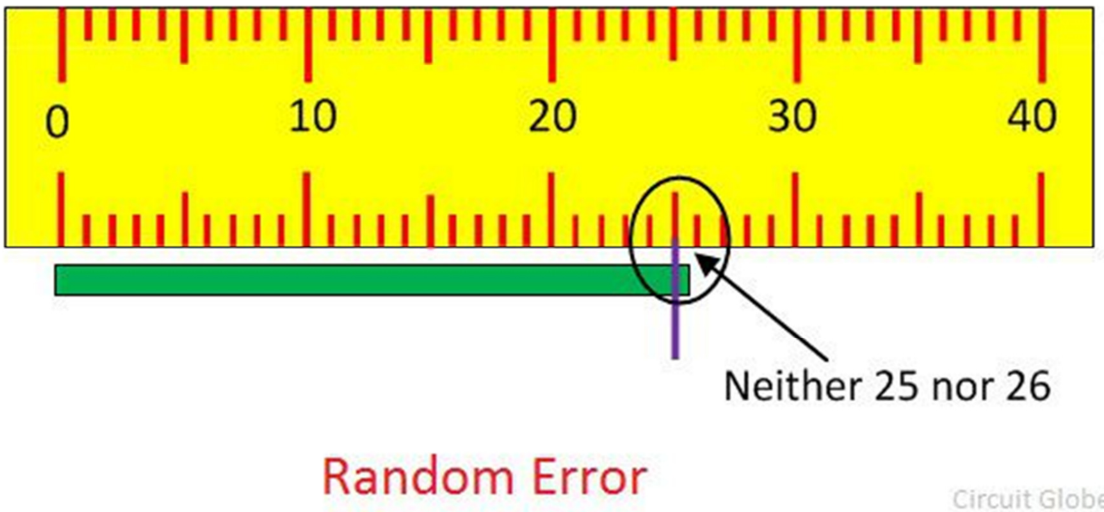
systematic errors → lead to predictable and consistent departures from the true value.

Random error:

Random error describes errors that fluctuate due to the unpredictability or uncertainty inherent in the concerned measuring process, or the variation in the quantity one is trying to measure [02].

Examples [01]:

01. Electronic noise in the electrical circuit involved in an electrical instrument,
02. Irregular changes in the heat loss rate from a solar collector owing to changes in the wind characteristics.



Fig, 01. Measuring the length of a rod the length of which is neither 25 mm nor 26 mm, but in between the two [04].

Propagation of Errors.

Propagation of error is the effect of variables' uncertainties or errors, on the uncertainty/error of a function based on them. The uncertainties involved in the variables due to measurement limitations (e.g., instrument precision) propagate due to the combination of variables in the function[03].

Example:

Let us consider the case of finding the cross sectional area, A, of a rectangular plate (Fig. 02). Let

L → length of the cross section,

B → Breadth of the same.

Let one can measure the above two dimensions as

L_0 and B_0, respectively.

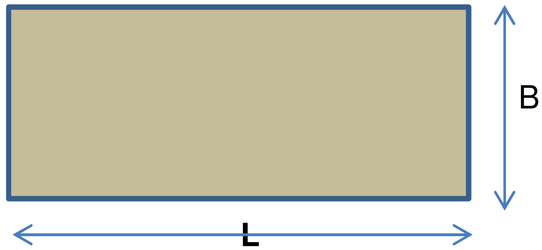


Fig. 02. Cross section of a rectangular plate.

Thus the measured cross sectional area, A0, can be written as

$$A_0 = L_0 B_0. \quad \ldots \quad \ldots. \quad \ldots. \quad (01)$$

Let us see how the uncertainties in L_0 and B_0 affect the uncertainty in A_0. Let the actual errors in L an B, respectively, are

$$\Delta L = L - L_0$$

and $$\Delta B = B - B_0.$$

An estimate of the error in the final result A_0 could be obtained by expanding A about the point (L_0, B_0) in a Taylor series. The first term of the expansion gives

$$A \cong A_0 + \Delta L\left(\frac{\partial A}{\partial L}\right)_B + \Delta B\left(\frac{\partial A}{\partial B}\right)_L \quad \ldots. \quad \ldots. \quad \ldots. \quad (02)$$

from which we can find out

$$\Delta A = A - A_0.$$

The terms in the parentheses are the partial derivatives of A, with respect to L and B, evaluated at the point L_0, B_0. The above expansion neglects the higher order terms in the Taylor expansion. If the errors are large, higher order terms should be taken into account according as the extent of the errors.

In our example, A = LB, and thus eq,(02) gives

$$\Delta A \cong B_0 \Delta L + L_0 \Delta B \quad \ldots \quad \ldots. \quad \ldots. \quad (03)$$

which entails that ΔA could be evaluated if one knew the uncertainties ΔL and ΔB.

We do not know, in general, actual errors in the determination of the dependent variables. However, we may be able to estimate the error in each measured quantities, or to estimate some characteristic, such as the standard deviation σ, of the probability distribution of the measured quantities.

Let us see how one can combine the standard deviations of the individual measurements to estimate the uncertainty in the result.

Let us suppose that x is a quantity which one wants to determine and that x is a function of at least two variables u and v. The characteristic of x could be determined from those of u and v. We write

$$x = f(u, v, ...) \quad ... \quad ... \quad ... \quad ... \quad ... \quad (04)$$

The most probable value of x may be assumed to be

$$\bar{x} = f(\bar{u}, \bar{v}, ...) \quad ... \quad ... \quad ... \quad ... \quad ... \quad (05)$$

Considering individual measurements u_i, v_i, etc into individual results xi, we write

$$x_i = f(u_i, v_i, ...) \quad ... \quad ... \quad ... \quad ... \quad ... \quad (06)$$

The variance which is the square of the standard deviation σ_x, is given by

$$\sigma_x^2 = \lim_{N \to \infty} \left[\frac{1}{N} \sum (x_i - \bar{x})^2 \right] \quad ... \quad ... \quad ... \quad ... \quad ... \quad (07)$$

Expressing the deviations $x_i - \bar{x}$ in terms of the deviations $u_i - \bar{u}$, $v_i - \bar{v}$, … of the observed parameters, we have

$$x_i - \bar{x} \cong (u_i - \bar{u}) \left(\frac{\partial x}{\partial u} \right) + (v_i - \bar{v}) \left(\frac{\partial x}{\partial v} \right) + ... , \quad ... \quad ... \quad ... \quad (08)$$

Each of the partial derivatives being evaluated with all the other variables fixed at their mean values.

The variables u, v, …, were actually measured. So we now express the variance σ_x^2 for x in terms of variances σ_u^2, σ_v^2, ..., for the measured variables u, v, …., as follows

$$\sigma_x^2 \cong \lim_{N\to\infty} \frac{1}{N}\Sigma\left[(u_i-\bar{u})\left(\frac{\partial x}{\partial u}\right)_{N\to\infty}+(v_i-\bar{v})\left(\frac{\partial x}{\partial v}\right)+\ldots\right]^2$$

$$\cong \lim_{N\to\infty} \frac{1}{N}\Sigma\left[(u_i-\bar{u})^2\left(\frac{\partial x}{\partial u}\right)^2+(v_i-\bar{v})^2\left(\frac{\partial x}{\partial v}\right)^2+2(u_i-\bar{u})(v_i-\bar{v})\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)+\ldots\right] \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (09)$$

Now,

$$\sigma_u^2 = \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(u_i-\bar{u})^2\right] \quad \ldots\ \ldots.\ \ldots..\ \ldots.\ \ldots.\ \ldots.\ \ldots\ (10a)$$

and

$$\sigma_v^2 = \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(v_i-\bar{v})^2\right] \quad \ldots\ \ldots.\ \ldots..\ \ldots.\ \ldots.\ \ldots.\ \ldots\ (10b)$$

σ_{uv}^2, the covarinces between the variables u and v, can similarly be expressed as

$$\sigma_{uv}^2 \cong \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(u_i-\bar{u})[(v_i-\bar{v})]\right] \quad \ldots\ \ldots..\ \ldots.\ \ldots.\ \ldots.\ \ldots\ (11)$$

With the above definitions, we can write

$$\sigma_x^2 \cong \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2+\sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2+\ldots+2\sigma_{uv}^2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)+\ldots \quad \ldots\ \ldots\ (12)$$

The above equation (12) is known as the *error propagation* equation.

If the fluctuations in the measured quantities u, v, … are uncorrelated, then in the limit of a large random selection of observations, the third term in eq. (12) may be expected to be vanished, and one can then write

$$\sigma_x^2 \cong \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2 + \ldots \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (13)$$

The above equation is, in general, used for determining the effects of measuring uncertainties in the final result.

References:

[01] https://www.physics.umd.edu/courses/Phys276/Hill/Information/Notes/ErrorAnalysis.html

[02] https://sciencing.com/difference-between-systematic-random-errors-8254711.html

[03] https://en.wikipedia.org/wiki/Propagation_of_uncertainty ; and the references therein

[04] circuit globe.com

Special statement: ***All the particulars provided here are as study material for the students and not for any other purpose. It has been sincerely tried to mention the concerned references wherever required.***