Economics Honours (Semester II)

Mathematical Economics -I

Partial Differentiation

The concept of Partial Differentiation is associated with the concept of partial functions. So in order to explain Partial Differentiation we need first to define partial functions.

***Definition of Partial Functions**

Given a function of n variables, we can introduce for each variable x_i, a family of partial functions where each partial function treats all other variables x_j, $j \neq i$, as constants. This can be represented as

$$x_i \longrightarrow f(\overline{x}_1, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$$

where the 'bars' over x_j, $j \neq i$, indicate we are treating them as constants in this mapping.

This notion of treating all other variables as constant corresponds to the economist's use of the phrase 'all other things being the same' or *ceteris paribus*. Just as we consider shifts in demand curves, etc. due to changes in some other variable, so we must also recognize that the partial function will shift as we consider different values of the other variables, thus we have a family of partial functions for each variable.

***Definition of Partial Differentiation**

Given a function $f(x)$ of n variables x_i , $i = 1,\ldots,n$, the derivatives of its associated partial functions are known as the partial derivatives of f and are denoted by $\delta f/\delta x_i$, or $f_i'(x)$.

Example: If $f(x_1, x_2) = p_1x_1 + p_2x_2$, one partial function can be written as

$$f(x_1, \overline{x}_2) = p_1x_1 + p_2\overline{x}_2$$

and the partial derivate of $f(x_1, x_2)$ with respect to x_1 is the derivative of $f(x_1, \overline{x_2})$

So $\delta f/\delta x_1 = f'(x_1, \overline{x_2}) = p_1$

Similarly $\delta f/\delta x_2 = f'(\overline{x_1}, x_2) = p_2$

***Partial Differentiation of Additive Separable Functions**

Statement: A function $f(x)$ of n variables x_i , i = 1,…,n, is said to be additively separable if it can be written in the form $f(x) = g_1(x_1) + \dots + g_n(x_n) = \Sigma\, g_i(x_i)$ for all i; i.e. it is the sum of separable functions of the n variables.

The partial differentiation of an additive function of the form $f(x) = \Sigma\, g_i(x_i)$ for all i, (i = 1,…,n), is given by

$\delta f/\delta x_i = dg_i / dx_i$, because the derivatives of $g_j(x_j)$, $j \neq i$, with respect to x_i will be zero.

Example: $f(x_1, x_2, x_3) = 4x_1 + 8x_2 + 2x_3$

Partial derivatives of f with respect to x_1, x_2 and x_3 are determined as follows:

$\delta f/\delta x_1 = dg_1 / dx_1 = 4$,

$\delta f/\delta x_2 = dg_2 / dx_2 = 8$,

$\delta f/\delta x_3 = dg_3 / dx_3 = 2$

***Partial Differentiation of Multiplicatively Separable Functions**

A function $f(x)$ of n variables x_i , i = 1,…,n, is said to be multiplicatively separable if it can be written in the form $f(x) = g_1(x_1) \times g_2(x_2) \times \dots \times g_n(x_n) = \Pi g_i(x_i)$ for all i, (i = 1, …,n). Here partial derivatives of f is given by $\delta f/\delta x_i$; x_j is held constant for all j, where $j \neq i$.

Example: If $f(x_1, x_2) = (x_1 . x_2)$, then $\delta f/\delta x_1 = x_2$ & $\delta f/\delta x_2 = x_1$.

***The Implicit Differentiation**

If x_1 is an implicit function of x_2 to x_n of the form f(x) = k then differentiation of the implicit function takes the form

$$\delta x_1/\delta x_i = -\frac{\delta f/\delta x_i}{\delta f/\delta x_1}$$

Example: Find dx_1/dx_2 when $f(x_1, x_2) = x_1^2 + x_2^2 = 9$, $x_1 > 0$, $x_2 > 0$

Using the implicit function rule for partial derivative

$$dx_1/dx_2 = -\frac{\delta f/\delta x_i}{\delta f/\delta x_1} = -(2x_2 / 2x_1) = -(x_2/x_1)$$

***Total Derivative**

If y = f(x) is a function of n variables x_i , i = 1,…,n, and each x_i is a function of the variable t, then total derivative of y with respect to t is given by

$$dy/dt = \frac{\delta f}{\delta x_1}\frac{\delta x_1}{\delta t} + \ldots + \frac{\delta f}{\delta x_n}\frac{\delta x_n}{\delta t} = f'(x)\,.\,(dx/dt)$$

Example: If $y = f(x_1,x_2) = x_1.x_2$, $x_1 = t^3$ and $x_2 = a + bt$, then

$$dy/dt = \frac{\delta f}{\delta x_1}\frac{dx_1}{dt} + \frac{\delta f}{\delta x_2}\frac{dx_2}{dt}$$

$$= x_2.\,3t^2 + x_1.b$$

$$= (a + bt)\,3t^2 + t^3.b$$

$$= 3at^2 + 4bt^3$$

*N.B. Explanations and all theoretical disposition have been accessed from the following references

References

1. Birchenhall, Chris and Grout, Paul. *Mathematics for Modern Economics*, (1987), Heritage Publishers, New Delhi.