Method of electrical images

When ~~...~~ a point charge or a distribution of charges is in front of a metallic surface then the potential and field in that region can be found out easily by the method of electrical images. This useful method is suggested by Lord Kelvin.

In this method of solution we are going to discuss, the actual electrification of the surface is replaced by one or more fictitious point charges in the region where the solution is not desired. The position and magnitude of imaginary charges are such that in the desired region Laplace's equation is satisfied.

" The fictitious point charges placed in the region where the field is not required and producing the same field in the desired region as with the actual electrification of the surface, are defined as electrical image."

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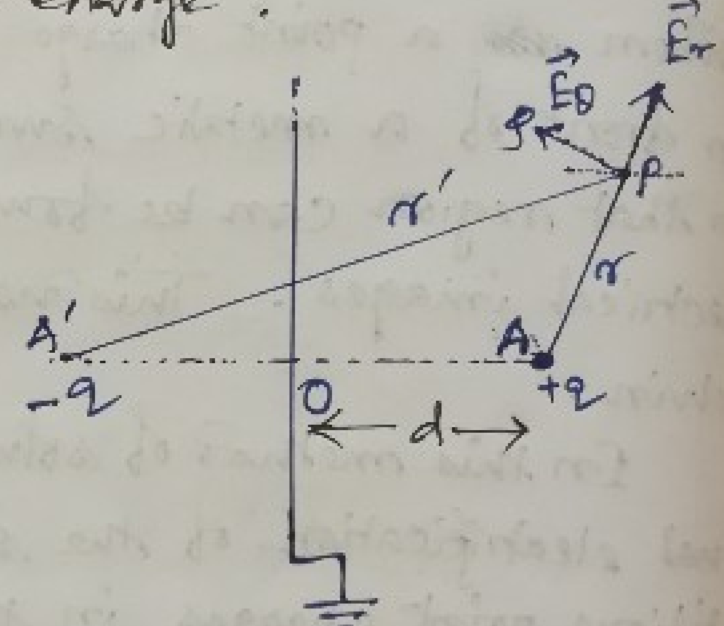
Point charge placed in front of an Earthed Conducting plane :-

Let us consider a point charge +q is at a distance d from an infinite conducting Earthed plane. Due to induction negative charge will appear over the surface. In the method of electrical image we have replaced these induced charge by an image charge.



~~Now~~ we have to find the potential and field at point P in that region where the actual charge is placed.

Now the potential at any point on the conducting surface is given by

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r'}\right) = 0 \quad \left[\text{Since the plate is grounded}\right]$$

So, obviously $r = r'$

Now since the point charge +q is placed at a distance d from the ~~plate~~ plane, so image charge -q must be placed at the same distance d in that region where the solution is not desired. Then all the boundary conditions are satisfied which are.

i) On the right hand side of the plane (where the solution is desired), Laplace's equation satisfies everywhere except at point A where the charge +q is located.

ii) The potential everywhere on the conducting plane must be zero.

iii) The potential at infinity must be zero.

I. Potential at point P :-

The potential at Point P

$$V_P = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r'}\right)$$

Now from the triangle A'PA, $r' = (r^2 + 4d^2 + 4dr\cos\theta)^{1/2}$

$$\therefore V_P = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{(r^2+4d^2+4dr\cos\theta)^{1/2}}\right]$$

II. Electric field at Point P :-

$$E_r = -\frac{\partial V_P}{\partial r}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r+2d\cos\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right], \text{ along } \vec{AP}$$

And $E_\theta = -\frac{1}{r}\frac{\partial V_P}{\partial \theta}$

$$= \frac{q}{4\pi\epsilon_0}\cdot\frac{2d\sin\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}, \text{ along } \vec{PQ}$$

(where PQ is perpendicular to AP)

Now the resultant electric field normal to the plane or parallel to AA' at P is

$$E_n = E_r\cos\theta - E_\theta\sin\theta$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2d\cos^2\theta}{(r^2+4d^2+4dr\cos\theta)^{3/2}} - \frac{2d\sin^2\theta}{(r^2+4d^2+4dr\cos\theta)^{3/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2d}{(r^2+4d^2+4dr\cos\theta)^{3/2}}\right]$$

$$E_n = \frac{q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2d}{r'^3}\right]$$

Therefore the normal component of electric field at the conducting plane can be found out by putting $r = r'$

$$\therefore E_n = -\frac{q}{4\pi\epsilon_0}\frac{2d}{r^3}$$

III Surface density of charge on the conducting plane :-

If σ be the surface charge density then from the Gauss's law of electrostatics the normal component of electric field

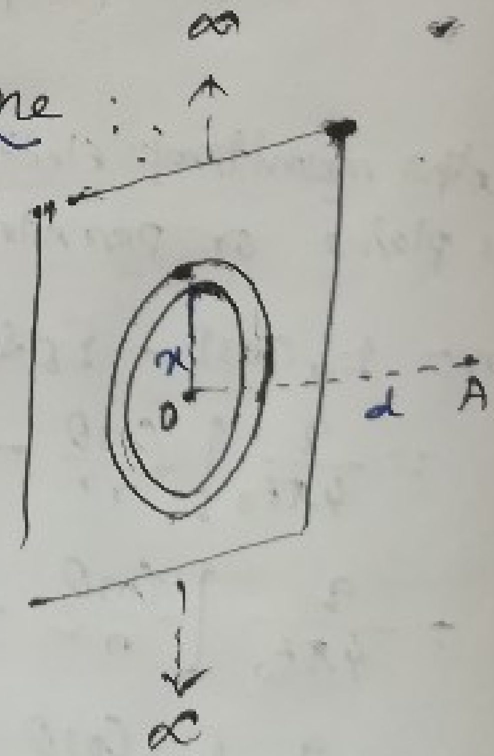
$$E_n = \sigma/\epsilon_0$$

$$\therefore \sigma = E_n \times \epsilon_0$$

$$= -\frac{qd}{2\pi r^3}.$$

From the above expression we can conclude that σ will be maximum when r is minimum ie the surface charge density is maximum near the foot of perpendicular OA .

IV. Total induced charge on the plane :



Let us consider ~~an small ring~~ a ring of radius x and thickness dx. Then charge contained by the elementary ring is

$$dq = \sigma ds$$

$$= \sigma \times 2\pi x dx.$$

So total charge induced on the plane is

$$= \int_0^\infty 2\pi\sigma x dx.$$

Now, $\sigma = -\dfrac{qd}{2\pi r^3}$ and $r = \sqrt{x^2+d^2}$

$\therefore$ Total charge induced

$$= -qd \int_0^{\infty} \frac{x\,dx}{(x^2+d^2)^{3/2}}$$

$$= -qd \left| -\frac{1}{(x^2+d^2)^{1/2}} \right|_0^{\infty}$$

$$= -qd \times \frac{1}{d}$$

$$= -q$$

So, total charge induced on the plane is $-q$.

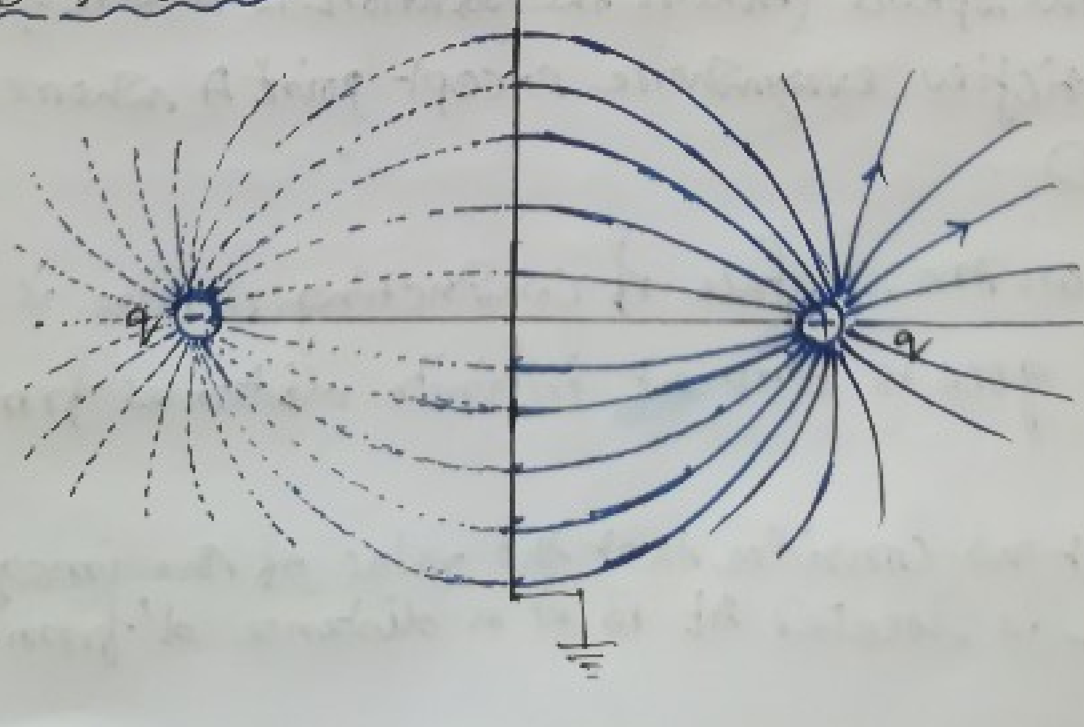
V. Force of attraction between the charge and the plane:

The force ~~of attraction~~ between $+q$ charge at A and the image charge $-q$ at A' is given by

$$F = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}$$

Negative sign indicates that the force is attractive.

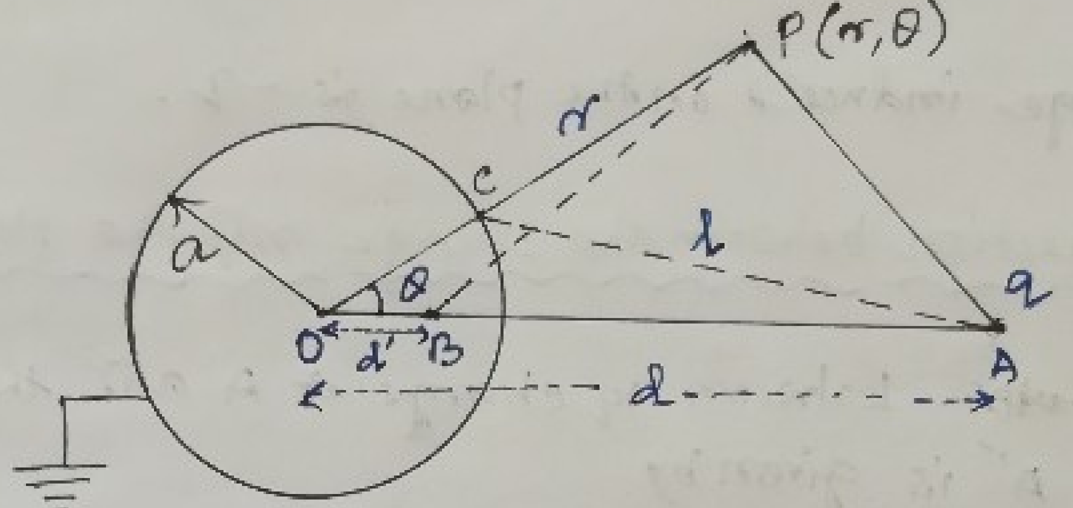
VI. Lines of force:



Point charge in front of a grounded conducting sphere :-

The method of electrical images can also be applied to calculate potential and field due to a point charge which is placed near a conducting sphere which has been earthed.

Let us consider a point charge q placed in front of a ~~sphere~~ earthed conducting sphere at A which is at a distance d from the centre of the sphere.



Now we have to find the magnitude and location of the image charge such that it satisfies the following boundary conditions —

i) Outside the sphere (where the solution is desired), Laplace's equation satisfies everywhere except point A where the charge q is located.

ii) Potential on the surface of conducting sphere is zero.

iii) Potential goes to zero at infinite distances from the sphere.

Now let us consider that the value of the image charge is q' which is located at B at a distance d' from the centre of the sphere. So the potential at a point P due to q and q' is

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(r^2+d^2-2rd\cos\theta)^{1/2}} + \frac{q'}{(r^2+d'^2-2rd'\cos\theta)^{1/2}}\right) \cdots (i)$$

On the surface of the sphere (ie at r=a) the potential is

$$V_s = \frac{1}{4\pi\epsilon_0}\left[\frac{q/a}{(1+d^2/a^2-2\frac{d}{a}\cos\theta)^{1/2}} + \frac{q'/d'}{(1+\frac{a^2}{d'^2}-2\frac{a}{d'}\cos\theta)^{1/2}}\right] = 0$$

Hence,

$$\frac{q/a}{(1+\frac{d^2}{a^2}-2\frac{d}{a}\cos\theta)^{1/2}} = -\frac{q'/d'}{(1+\frac{a^2}{d'^2}-2\frac{a}{d'}\cos\theta)^{1/2}}$$

To satisfy the above we can write

$$q/a = -q'/d' \quad \cdots\cdots\cdots (ii)$$

and $$1+\frac{d^2}{a^2}-2\frac{d}{a}\cos\theta = 1+\frac{a^2}{d'^2}-2\frac{a}{d'}\cos\theta \quad \cdots (iii)$$

So, $\frac{d}{a} = \frac{a}{d'}$ (coefficient of $\cos\theta$)

ie $d' = \frac{a^2}{d}$

$\therefore$ from eqn (ii)

$$q' = -\frac{q}{a}d' = -\frac{q}{a}\times\frac{a^2}{d}$$

$$\therefore q' = -q\frac{a}{d}$$

I. Potential and field at P:

substituting the values of q' and d' in eqn (i) we get

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{a/d}{(r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta)^{1/2}}\right]$$

$$\therefore V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{a}{(r^2d^2+a^4-2rda^2\cos\theta)^{1/2}}\right] \quad \text{(iv)}$$

So, the r component of electric field at P is

$$E_r = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{r-d\cos\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{ad(rd-a^2\cos\theta)}{(r^2d^2+a^4-2rda^2\cos\theta)^{3/2}}\right]$$

The normal electric field at the point C on the surface can be determined by putting $r=a$

Then $E_n = \dfrac{q}{4\pi\epsilon_0}\dfrac{(a^2-d^2)}{a(a^2+d^2-2ad\cos\theta)^{3/2}}$

$$= \frac{q}{4\pi\epsilon_0 l^3}\left(a - d^2/a\right) \quad \left[\because l = AC = (a^2+d^2-2ad\cos\theta)^{1/2}\right] \quad \text{(v)}$$

Like E_r, E_θ can also be found ~~by~~ out ~~by~~.

where $E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta}$

The value of E_θ on the surface of spherical conductor is zero.

(II). Surface charge density on the surface due to induced charge on the sphere:

If σ be the surface charge density then from Gauss's law of electrostatics the normal component of electric field

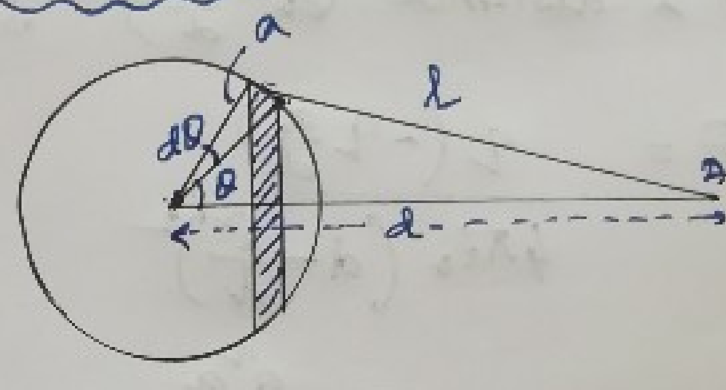
$E_n = \sigma/\epsilon_0$

$\therefore \sigma = \epsilon_0 E_n$

$$= \frac{q}{4\pi l^3}\left(a - d^2/a\right) = -\frac{q}{4\pi l^3}\left(\frac{d^2-a^2}{a}\right) \quad [d>a] \quad \text{(vi)}$$

From the above expression, the surface charge density is maximum when l is minimum ie when $\theta = 0$ ie when $l_{min} = d-a$. And the surface charge density is minimum when l is maximum ie when $\theta = 180°$ then $l_{max} = d+a$.

(iii) Total charge induced on the sphere:



Let us consider a ring within the sphere of radius $a\sin\theta$ and width $a\,d\theta$. Now the charge induced within this ring is

$$dq = 2\pi a \sin\theta \times a\,d\theta \times \sigma$$ where σ is surface charge density.

$$= 2\pi a^2 \sigma \sin\theta\, d\theta$$

$$= -\frac{q}{2l^3}\, a(d^2-a^2)\sin\theta\, d\theta.$$

Now, $l = (a^2+d^2-2ad\cos\theta)^{1/2}$

$\therefore l^2 = a^2+d^2-2ad\cos\theta$

So, $l\,dl = ad\sin\theta\, d\theta$.

$\therefore \sin\theta\, d\theta = \frac{l\,dl}{ad}$

So, total charge induced within the sphere is

$$= -\int_{d-a}^{d+a} \frac{q}{2l^3}\, a(d^2-a^2) \times \frac{l\,dl}{ad}$$

$$= -\frac{q(d^2-a^2)}{2d}\int_{d-a}^{d+a}\frac{dl}{l^2} = \frac{q(d^2-a^2)}{2d}\left[\frac{1}{l}\right]_{d-a}^{d+a}$$

$$= -q\frac{a}{d}.$$

IV. The force between the conducting sphere and the point charge at A :-

This actually means the force between point charge $+q$ and the image charge $-q\frac{a}{d}$ which are separated by a distance $\left(d-\frac{a^2}{d}\right)$. So,

$$F = \frac{q\left(-q\frac{a}{d}\right)}{4\pi\epsilon_0\left(d-\frac{a^2}{d}\right)^2}$$

$$= -\frac{q^2 a}{4\pi\epsilon_0 d\left(d-\frac{a^2}{d}\right)^2}$$

The negative sign indicates that the force is attractive.

V. Lines of force :-