Economics Honours (Semester VI)
Basic Econometrics
Properties of Least-Squares Estimators

**Box 1: Derivation of Least-Squares Estimates**

Given the two variable estimated regression equation as:

$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$, for i=1,2,…,n (1)

where $X_i$ = independent variable, $Y_i$ = dependent variable, $\hat{\beta}_1$ & $\hat{\beta}_2$= the estimated coefficients and $\hat{u}_i$ = the estimated residuals.

$Y_i = \hat{Y}_i + \hat{u}_i$ (2)

Where $\hat{Y}_i$ is the estimated value of $Y_i$.

Therefore, $\hat{u}_i = Y_i - \hat{Y}_i$

$= Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$ (3)

The summation of the squared residuals is given by

$\Sigma \hat{u}_i^2 = \Sigma (Y_i - \hat{Y}_i)^2 = \Sigma (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$ (4)

By the criterion of Least Squares, differentiating (4) partially with respect to $\hat{\beta}_1$ and $\hat{\beta}_2$, we obtain

$\partial (\Sigma \hat{u}_i^2)/\partial \hat{\beta}_1 = -2\, \Sigma(Yi - \hat{\beta}_1 - \hat{\beta}_2 Xi) = -2\, \Sigma \hat{u}_i$ (5)

$\partial (\Sigma \hat{u}_i^2)/\partial \hat{\beta}_2 = -2\, \Sigma(Yi - \hat{\beta}_1 - \hat{\beta}_2 Xi)\, Xi = -2\, \Sigma \hat{u}_i\, Xi$ (6)

Setting these normal equations 5. and 6. equal to zero, and after algebraic simplification and manipulation, we get the estimators as

$\hat{\beta}_2 = \dfrac{\Sigma x_i\, y_i}{\Sigma x_i^2}$ (7)

$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$ (8)

Notes:
Here $x_i = X_i - \bar{X}$ (where $\bar{X}$ means mean of $X_i$),
$y_i = Y_i - \bar{Y}$ (where $\bar{Y}$ means mean of $Y_i$)
$\Sigma x_i^2 = \Sigma(X_i - \bar{X})^2 = \Sigma X_i^2 - 2\Sigma X_i \bar{X} + \Sigma \bar{X}^2 = \Sigma X_i^2 - 2\bar{X}\Sigma X_i + \Sigma \bar{X}^2 = \Sigma X_i^2 - n\bar{X}^2$
since $\bar{X}$ is a constant; $\Sigma X_i = n\bar{X}$; $\Sigma \bar{X}^2 = n\bar{X}^2$.
$\Sigma x_i y_i = \Sigma x_i(Y_i - \bar{Y}) = \Sigma x_i Y_i - \bar{Y}\Sigma xi = \Sigma x_i Y_i - \bar{Y}\Sigma(X_i - \bar{X}) = \Sigma x_i Y_i$ since $\bar{Y}$ is a constant; $\Sigma x_i = 0$; $\Sigma y_i = 0$.
Similarly, $\Sigma x_i y_i = \Sigma X_i y_i$.

Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

**Box 2: Properties of Least-Square Estimators ($\beta^{\wedge}_1$ & $\beta^{\wedge}_2$)**

With the assumption that the error term '$u_i$' follows the normal distribution, the OLS estimators have the following statistical properties:

1. The estimators $\beta^{\wedge}_1$ & $\beta^{\wedge}_2$ are **linear functions** of the random variable that is the dependent variable $Y_i$ and are **normally distributed**.
2. The estimators $\beta^{\wedge}_1$ & $\beta^{\wedge}_2$ are **unbiased**. This means **$E(\beta^{\wedge}_1) = \beta^{\wedge}_1$ & $E(\beta^{\wedge}_2) = \beta^{\wedge}_2$.**
3. The estimators $\beta^{\wedge}_1$ & $\beta^{\wedge}_2$ have **minimum variance** or least variance. This means the unbiased estimators having minimum variance are **efficient estimators**.
4. They have **consistency** which means as the sample size increases, the estimators converge to their true population value

Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

# *The Gauss Markov Theorem

Given the assumptions of Classical Linear regression Model, the OLS estimators are BLUE, that is, they are *Best Linear Unbiased Estimators* possessing minimum variance.

* Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

**Box 3: Linearity Property of Least-Squares Estimators**

From Equation (7) in Box 1 we have $\hat{\beta}_2 = (\Sigma x_i Y_i)/(\Sigma x_i^2) = \Sigma k_i Y_i$ …………………(9)

where $k_i = xi/(\Sigma x_i^2)$

which shows that $\hat{\beta}_2$ is a linear estimator because it is a linear function of Y; actually it is a weighted average of $Y_i$ with $k_i$ serving as the weights. It can similarly be shown that $\hat{\beta}_1$ too is a linear estimator.

Note:

Incidentally, note these properties of the weights $k_i$:

1. Since the $X_i$ are assumed to be nonstochastic, the $k_i$ are nonstochastic too.
2. $\Sigma k_i = 0$.
3. $\Sigma k_i^2 = 1/ \Sigma x_i^2$ .
4. $\Sigma k_i x_i = \Sigma k_i X_i = 1$. These properties can be directly verified from the definition of $k_i$.

For example,

$\Sigma ki = \Sigma(x_i/ \Sigma x_i^2) = (1/ \Sigma x_i^2). \Sigma x_i^2$ since for a given sample $\Sigma x_i^2$ is known

$= 0,$ since $\Sigma x_i$, the sum of deviations from the mean value, is always zero

Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

**Box 4: Unbiasedness Properties of Least-Squares**

Suppose the Population Regression Function is $Y_i = \beta_1 + \beta_2 X_i + u_i$.

Let us substitute this function into Equation (9) of Box 3 to obtain

$\hat{\beta}_2 = \Sigma k_i(\beta_1 + \beta_2 X_i + u_i)$

$= \beta_1 \Sigma k_i + \beta_2 \Sigma k_i X_i + \Sigma k_i u_i$ ………………………………………………………. (10)

$= \beta_2 + \Sigma k_i u_i$ {where use is made of the properties of $k_i$ noted earlier in Box 3.}.

Now taking expectation of (10) on both sides and noting that $k_i$, being nonstochastic, can be treated as constants, we obtain

$E(\hat{\beta}_2) = \beta_2 + k_i E(u_i) = \beta_2$ ………………………………………………………….(11)

since $E(u_i) = 0$ by assumption. Therefore, $\hat{\beta}_2$ is an unbiased estimator of $\beta_2$. Likewise, it can be proved that $\hat{\beta}_1$ is also an unbiased estimator of $\beta_1$.

Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

**Box 5: Minimum-Variance Property of Least-Squares Estimators**

It has been shown in Box 3 and Box 4, that the least-squares estimator $\hat{\beta}_2$ is linear as well as unbiased (this holds true of $\hat{\beta}_1$ too). To show that these estimators are also minimum variance in the class of all linear unbiased estimators, consider the least-squares estimator

$\hat{\beta}_2 = \Sigma k_i Y_i$; where $k_i = (X_i - \bar{X}) / \Sigma(X_i - \bar{X})^2 = (x_i / \Sigma x_i^2)$ …………………………..(12)

which shows that $\hat{\beta}_2$ is a weighted average of the Y's, with $k_i$ serving as the weights.

Let us define an alternative linear estimator of $\beta_2$ as follows: $\beta^*_2 = \Sigma w_i Y_i$ …...............(13)

where $w_i$ are also weights, not necessarily equal to $k_i$. Now

$E(\beta^*_2) = \Sigma w_i \, E(Y_i)$

$= \Sigma w_i \, (\beta_1 + \beta_2 X_i)$ ………………………………………………………….(14)

$= \beta_1 \, \Sigma w_i + \beta_2 \, \Sigma w_i \, X_i$

Therefore, for $\beta^*_2$ to be unbiased, we must have

$\Sigma w_i = 0$ ……………………………………………………………………………(15)

and $\Sigma w_i \, X_i = 1$ ……………………………………………………………………….(16)

Also, we may write $\text{var}(\beta^*_2) = \text{var} \, \Sigma w_i Y_i$

$= \Sigma w_i^2 \, \text{var} Y_i$ [*Note: $\text{var} Y_i = \text{var} \, u_i = \sigma \, 2$*]

$= \sigma^2 \Sigma \, w_i^2$ [*Note: $\text{cov}(Yi, Yj) = 0 \; (i \neq j)$*]

$= \sigma^2 \, \Sigma(w_i - x_i / \Sigma x_i^2 + x_i / \Sigma x_i^2)^2$ [Note the mathematical trick]

$= \sigma^2 \, \Sigma(w_i - x_i / \Sigma x_i^2)^2 + \sigma^2 \, (\Sigma x_i)^2 / (\Sigma x_i^2)^2 - 2 \, \sigma^2 \, \Sigma(w_i - x_i / \Sigma x_i^2)(x_i / \Sigma x_i^2)$

$= \sigma^2 \, \Sigma(w_i - x_i / \Sigma x_i^2)^2 + \sigma^2 (1 / \Sigma x_i^2)$ ……………………………………….(17)

[Because the last term in the next to the last step drops out]

The last term in (17) is constant, the variance of $(\beta^*_2)$ can be minimized only by manipulating the first term. If we let $w_i = (x_i / \Sigma x_i^2)$

Eq. (17) reduces to $\text{var}(\beta^*_2) = \sigma^2 / \Sigma x_i^2 = \text{var}(\hat{\beta}_2)$ ………………………………..(18)

In words, with weights $w_i = k_i$, which are the least-squares weights, the variance of the linear estimator $\beta^*_2$ is equal to the variance of the least squares estimator $\hat{\beta}_2$; otherwise $\text{var}(\beta^*_2) > \text{var}(\hat{\beta}_2)$. To put it differently, if there is a minimum-variance linear unbiased estimator of $\beta_2$, it must be the least-squares estimator. Similarly it can be shown that $\hat{\beta}_1$ is a minimum variance linear unbiased estimator of $\beta_1$.

Source: Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).

**Box 6: Consistency of Least-Squares Estimators**

We have shown that, in the framework of the classical linear regression model, the least-squares estimators are unbiased (and efficient) in any sample size, small or large. But sometimes, an estimator may not satisfy one or more desirable statistical properties in small samples. But as the sample size increases indefinitely, the estimators possess several desirable statistical properties. These properties are known as the **large sample**, or **asymptotic properties**. Here we discuss one large sample property, namely, the property of consistency. For the two-variable model we have already shown that the OLS estimator $\hat{\beta}_2$ is an unbiased estimator of the true $\beta_2$. Now we show that $\hat{\beta}_2$ is also a consistent estimator of $\beta_2$. A sufficient condition for consistency is that $\hat{\beta}_2$ is unbiased and that its variance tends to zero as the sample size n tends to infinity.

Since we have already proved the unbiasedness property, we need only show that the variance of $\hat{\beta}_2$ tends to zero as n increases indefinitely. We know that

$$\operatorname{var}(\hat{\beta}_2) = \sigma^2 / \Sigma x_i^2 = \frac{\sigma^2/n}{\Sigma x_i^2/n} \quad \ldots\ldots(19)$$

By dividing the numerator and denominator by n, we do not change the equality.

Now

$$\lim_{n \to \infty} \operatorname{var}(\hat{\beta}_2) = \lim_{n \to \infty} \frac{\sigma^2/n}{\Sigma x_i^2/n} = 0 \quad \ldots\ldots(20)$$

where use is made of the facts that (1) the limit of a ratio quantity is the limit of the quantity in the numerator to the limit of the quantity in the denominator; (2) as n tends to infinity, $\sigma^2/n$ tends to zero because $\sigma^2$ is a finite number; and $[\Sigma x_i^2/n] \neq 0$ because the variance of X has a finite limit because of assumption in CLRM. The upshot of the preceding discussion is that the OLS estimator $\hat{\beta}_2$ is a consistent estimator of true $\beta_2$. In like fashion, we can establish that $\hat{\beta}_1$ is also a consistent estimator. Thus, in repeated (small) samples, the OLS estimators are unbiased and as the sample size increases indefinitely the OLS estimators are consistent.

Source: Source: Gujarati, D. N. Porter, D.C., Gunasekar, S. (2009), *Basic econometrics*. (Fifth ed.) McGraw-Hill Education (India).