Differential Calculus

***Increment**

Suppose we have a variable x and the value of a variable x increases or decreases. The change in the value of x is the difference obtained by subtracting the first value from the second value. This change or difference is called increment. An increment in x is denoted by Δx (read as delta x). The sign of increment may be positive or negative depending upon the increase or decrease of x.

Suppose we have a function, $y = f(x)$, where x is said to be the independent variable and y the dependent variable.

An increase in x say Δx leads to an increase in y also say Δy. This is shown as follows

$$y + \Delta y = f(x + \Delta x)$$

$$\text{i.e. } \Delta y = f(x+\Delta x) - y$$

$$\text{i.e. } \Delta y = f(x+\Delta x) - f(x)$$

Example: Let $y = x^3$

Suppose x increases from 3 to 3.3, then $\Delta x = 0.3$. Hence correspondingly y also increases from 27 to 35.937. Thus $\Delta y = 8.937$

On the other hand suppose x decreases from 3 to 2, $\Delta x = -1$. Hence correspondingly y also decreases from 27 to 8. Thus $\Delta y = -19$.

Here it must be pointed out that Δx is also represented as 'h' and Δy is also represented as 'k'.

***Definition of Derivative**

Let $y = f(x)$ be a finite and single valued function defined in any interval of x and assume x to have a particular value in the interval. Let Δx (or h)be the increment of x and let Δy (or k) = $f(x+\Delta x) - f(x)$ be the corresponding increment of y. If the ratio $\Delta y/\Delta x$ of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called differential coefficient or derivative of f(x) (or Y) for the particular value of x, and is denoted by

$f'(x)$, $\frac{d\{f(x)\}}{dx}$, $\frac{dy}{dx}$, or, $D\{f(x)\}$.

Thus symbolically, the differential coefficient or derivative of $y = [f(x)]$ with respect to x (for any particular value of x) is

$$f'(x) \text{ or } \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h} \text{, provided this limit exists.}$$

***First and Second Order Derivatives**

The **first order derivative** or the first order differentiation of a function measures the rate of change of the dependent variable with respect to the independent variable. Let us address the following consumption function: C = C(Y), where C indicates consumption and is the dependent variable and Y indicates income the independent variable. If change in consumption is denoted by ΔC and change in income by ΔY, then $\Delta C/\Delta Y = dC/dY$ measures the rate of change in consumption with respect to Y i.e. the change in consumption due to one unit change in income. This is termed MPC or Marginal Propensity to Consume in Economics. As we can find a derivative of the aforesaid consumption function, the function is called a differentiable function. This first order derivative of a function is also defined as the slope of the function. This is explained diagrammatically with the above example on consumption function.

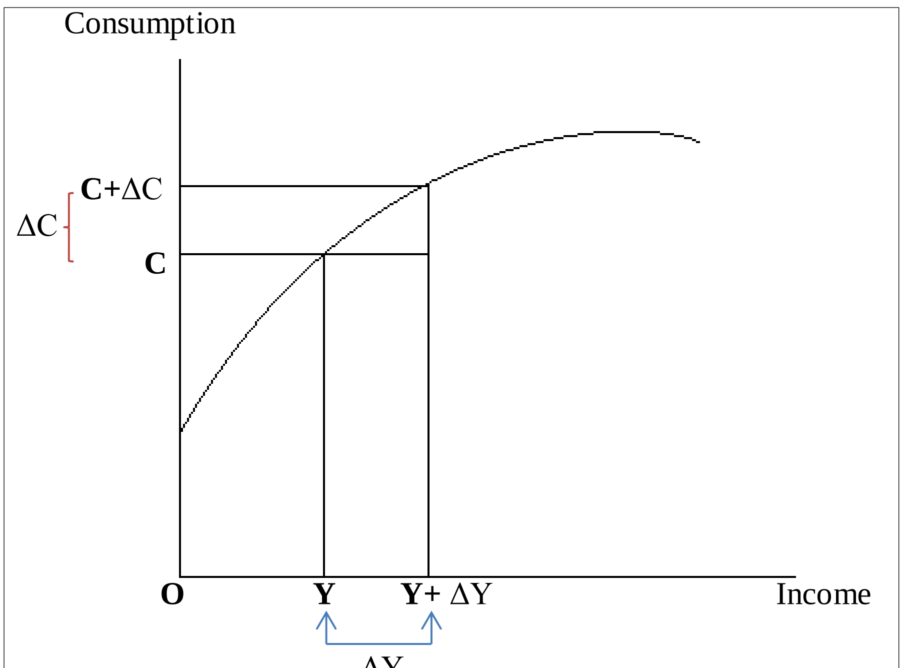

Fig. 1: A non-linear consumption function

A non-linear consumption function is drawn in Fig 1. The change in consumption is given by

$\Delta C = (C+ \Delta C) – \Delta C$. The change in income is given by $\Delta Y = (Y+ \Delta Y) – \Delta Y$. The ratio $\Delta C/\Delta Y$ or dC/dY determines the slope of the non-linear consumption function.

If we express the function of a curve as $y=f(x)$, then geometrically the slope of the curve stated as dy/dx means the value of the tangent drawn at a particular point on the curve. This is shown in Fig.2

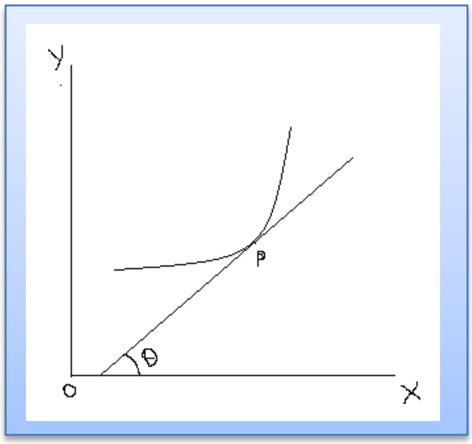

Fig.2: At point P a tangent touches a curve

Fig.2: shows that a tangent that touches a curve at point P meets the x axis at an angle of θ. Hence theoretically the value of the slope of the curve at P is given by tan θ or $dy/dx = \tan θ$. So slope at point $P = dy/dx = \tan θ$.

In the function $y=f(x)$, if for any given value of x, $dy/dx > 0$, then the function is increasing and the curve drawn is upward rising and if $dy/dx < 0$, then the function is decreasing and the curve is downward sloping. Further for any given value of x, if $dy/dx = 0$, then we determine the optimum point on the curve.

The **second order derivative** is defined as the differentiation of the first order derivative. Given a differentiable function $y = f(x)$, the first order derivative is itself a function. If the derivative function is stated as $dy/dx = f'(x)$, the second order derivative of $f(x)$ is expressed as the derivative of the derivative function and denoted by $d^2y/dx^2 = f''(x)$. The second order derivative d^2y/dx^2 is read as 'dee two y by dee x squared' and $f''(x)$ is read as 'f double prime of x'. This means if the first order derivative measures the rate of change of one variable in response to a change in another variable, then second order derivative measures the rate of this rate of change.

In the function $y=f(x)$, if for any given value of x, $d^2y/dx^2 > 0$, then the function has a local minimum and if $d^2y/dx^2 < 0$, then the function has a local maximum.

***To find the local optimum of a function**

To find if for x= x*, the given unconstrained function y = f(x)is the local optimum point or not, we need to fulfill the following conditions:

(a) A necessary condition for x* to be a local optimum of f(x) is that x* be a stationary point of f(x), i.e., dy/dx= f ’(x) = 0

(b) A sufficient condition for a stationary point, x*, to be a local maximum is d^2y/dx^2 = f ”(x) < 0 or to be a local minimum is or d^2y/dx^2 = f ”(x) > 0.

***List of some standard derivatives:**

Simple Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}c = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}cx = c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|}, \quad x \neq 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^c = cx^{c-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^c}\right) = -cx^{-(c+1)} = -\frac{c}{x^{c+1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) = x^{1/2} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

Exponential and Logarithmic Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x ln(a)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^x) = x^x(1 + lnx)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} ln(x) = \frac{1}{x}, \quad x > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x} log_a(x) = \frac{1}{xln(a)}$$

Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x} sin(x) = cos(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} cos(x) = -sin(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} tan(x) = sec^2(x) = \frac{1}{cos^2(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} sec(x) = sec(x)tan(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} csc(x) = -csc(x)cot(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} cot(x) = -csc^2(x)$$

Inverse Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x} sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

*N.B. Explanations of all the theories have been taken from the following references:

References

Das, B.C. & Mukherjee, B. N. (1949) *Differential Calculus for Degree Course.(1988)* Rajendranath Dhur for U.N. Dhur & Sons, Private Ltd, Calcutta

Birchenhall, Chris and Grout, Paul. *Mathematics for Modern Economics*, (1987), Heritage Publishers, New Delhi.

List of derivatives | Wyzant resources. (n.d.). Wyzant: Find Private Tutors at Affordable Prices, In-Person & Online. https://www.wyzant.com/resources/lessons/math/calculus/differentiation/list_of_derivatives