Some Special Integrals

(Core Course-03: Mathematical methods of Physics-II)

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Some Special Integrals: Beta and Gamma Functions and Relation between them. Expression of integrals in terms of Gamma functions. Error function (Probability integrals). (5L)

Beta function:-

The beta function is defined as,

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \quad \text{——— (1)}$$

which Converges when m and n are positive integers i.e. when $m>0$ and also $n>0$.

Let us Substitute, $x = (1-y)$ in equation (1).

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(1-1+y)^{n-1}\,d(1-y)$$

$$= +\int_0^1 (1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n,m)$$

$$\text{Thus } \boxed{\beta(m,n) = \beta(n,m)} \quad \text{——— (2)}$$

It Shows that, β-function is Symmetrical in m & n.

An important form for $\beta(m,n)$ Can be obtained by putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$ in equation (1).

Then we have,

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot \sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\,d\theta} \quad \text{—(3)}$$

This function is also Called Euler's integral of the first Kind.

Gamma function :- The Gamma function is defined as,

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad (n>0)} \qquad \text{--- (4)}$$

This integral is also known as Euler's integral of the Second kind. It follows immediately from the definition that,

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = \left| -e^{-x} \right|_0^\infty = -(e^{-\infty} - e^{-0})$$

or, $$\boxed{\Gamma(1) = -(0-1) = 1} \qquad \text{--- (5)}$$

Reduction or recursion formula for $\Gamma(n)$:

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\, dx$$

$$= \int_0^\infty e^{-x} x^n\, dx$$

(Integrating by parts)

$$= \cancel{x^n(-e^{-x})}\Big|_0^\infty - \int_0^\infty n x^{n-1}(-e^{-x})\, dx$$

$$= n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$\Rightarrow \boxed{\Gamma(n+1) = n\Gamma(n)} \qquad \text{--- (6)}$$

Equation (6) gives, $\Gamma(n) = \frac{1}{n}\Gamma(n+1)$ which tends to ∞ as $n \to 0$ and negative integers.

Value of $\Gamma(n)$ in terms of factorial :

Using $\Gamma(n+1) = n\Gamma(n)$ Successively, we get,

$$\Gamma(2) = 1\,\Gamma(1) = 1!$$
$$\Gamma(3) = 2\,\Gamma(2) = 2!$$
$$\Gamma(4) = 3\,\Gamma(3) = 3!$$
$$\cdots \quad \cdots \quad \cdots$$
$$\boxed{\Gamma(n+1) = n!} \qquad \text{--- (7)}$$

This shows that the gamma function can be regarded as a generalization of the elementary factorial function.

Value of $\Gamma(1/2)$:

As we know, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\therefore \Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} dx$$

Let, $x = y^2$ so that $dx = 2y\,dy$ and therefore, we have,

$$\therefore \Gamma(1/2) = \int_0^\infty e^{-y^2} y^{-1} \cdot 2y\,dy = 2\int_0^\infty e^{-y^2} dy$$

Again, it can be written that, $\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx$

$$\therefore \Gamma(1/2)\cdot\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Switching over to polar co-ordinates (r,θ) such that,

$x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

$$\therefore \Gamma(1/2)\cdot\Gamma(1/2) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\cdot\pi/2 \int_0^\infty e^{-r^2}\, r\,dr$$

$$\Rightarrow \Gamma(1/2)\cdot\Gamma(1/2) = 2\pi\left[(-1/2)\, e^{-r^2}\right]_0^\infty = \pi$$

$$\therefore \boxed{\Gamma(1/2) = \sqrt{\pi}} \quad \text{———} \quad (8)$$

*[1] $\Gamma(+1/2) = \Gamma(1-1/2) = (-1/2)\Gamma(-1/2)$ using, $\Gamma(n+1) = n\Gamma(n)$

or, $\Gamma(-1/2) = (-2)\Gamma(1/2) = -2\sqrt{\pi}$ (using, $\Gamma(1/2) = \sqrt{\pi}$)

*[2] $\Gamma(+3/2) = \Gamma(1+1/2) = 1/2\,\Gamma(1/2) = \sqrt{\pi}/2$

*[3] $\Gamma(-1/2) = \Gamma(1-3/2) = (-3/2)\Gamma(-3/2)$ (using, $\Gamma(-1/2) = -2\sqrt{\pi}$)

$$\therefore \Gamma(-3/2) = \frac{\Gamma(-1/2)}{(-3/2)} = -\frac{2\sqrt{\pi}}{(-3/2)} = \frac{4\sqrt{\pi}}{3}$$

Relation between beta and gamma functions :-

From the definition we have, $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt$

putting, $t = x^2$

So that, $dt = 2x\,dx$

$$\therefore \Gamma(m) = \int_0^\infty e^{-x^2} x^{2m-2} \cdot 2x\,dx = 2\int_0^\infty e^{-x^2} \cdot x^{2m-1}\,dx \quad \text{---(9)}$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2} \cdot y^{2n-1}\,dy$

$$\therefore \Gamma(m)\cdot\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \cdot 2\int_0^\infty e^{-y^2} \cdot y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1} \cdot y^{2n-1}\,dx\,dy \quad \text{---(10)}$$

(As, the limits of integration are constant)

Let us now change to polar co-ordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$. To cover the region in (10) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus equation (10) becomes,

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} \cdot (r\sin\theta)^{2n-1}\, r\,dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}\, r^{(2m-1+2n-\cancel{1}+\cancel{1})} \cdot \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, dr\,d\theta$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr\right] \quad \text{---(11)}$$

or, $\Gamma(m)\cdot\Gamma(n) = \beta(m,n) \times \Gamma(m+n)$ (using equation (9))

$$\text{or,} \quad \boxed{\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}} \quad \text{———— (12)}$$

Equation (12) gives the required important relation between beta and gamma functions.

Expressions of integrals in terms of Gamma functions :-

*. Evaluate the integral: $\int_0^{\pi/2} \sin^p x \cos^q x \, dx$

→ $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\int_0^{\pi/2} 2\sin^p x \cos^q x \, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 2\sin^{2\cdot\frac{1}{2}(p+1)-1} x \cos^{2\cdot \frac{1}{2}(q+1)-1} x \, dx$$

$$= \frac{1}{2}\beta\left[\tfrac{1}{2}(p+1), \tfrac{1}{2}(q+1)\right]$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Rightarrow \boxed{\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$$

*. Two important integrals : Dirichlet's integral†

1. $$\iint_D x^{l-1} y^{m-1} \, dx\, dy = \frac{\Gamma(l)\cdot\Gamma(m)}{\Gamma(l+m+1)} h^{l+m}$$

where, D is the domain $x \geq 0$, $y \geq 0$ and $x+y \leq h$

†2. $$\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx\,dy\,dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

where, V is the region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$.

Where integral 2 is known as Dirichlet's integral.

Proof of 1 : We put $x/h = X$ and $y/h = Y$. Then the integral reduces to,
$$\iint_{D'} (hX)^{l-1}(hY)^{m-1} h^2 \, dX\,dY, \quad \left\{\begin{array}{l}\text{where, } D' \text{ is the domain}\\ X \geq 0, Y \geq 0, X+Y \leq 1\end{array}\right.$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1} Y^{m-1} \, dX\,dY = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m \, dX = \frac{h^{l+m}}{m}\beta(l, m+1) = \frac{h^{l+m}}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)}$$

or, $$\boxed{\iint_D x^{l-1} y^{m-1} \, dx\,dy = \frac{\Gamma(l)\cdot\Gamma(m)}{\Gamma(l+m+1)} h^{l+m}} \text{; using } \Gamma(m+1) = m\Gamma(m)$$

Proof of 2:- We put $y = z \leq 1-x$ $(= h \text{ say})$, so that the integral reduces to

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dz\,dy\,dx$$

$$= \int_0^1 x^{l-1}\left[\int_0^h\int_0^{h-y} y^{m-1} z^{n-1}\, dz\,dy\right] dx$$

$$= \int_0^1 x^{l-1}\, \frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n+1)}\, h^{m+n}\, dx \;; \quad \text{using integral 1.}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\int_0^1 x^{l-1}(1-x)^{m+n}\, dx$$

$$= \frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n+1)}\,\beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n+1)}\cdot\frac{\Gamma(l)\,.\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(m)\,.\,\Gamma(n)\,.\,\Gamma(l)}{\Gamma(l+m+n+1)}$$

$$\therefore \boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma(l)\,.\,\Gamma(m)\,.\,\Gamma(n)}{\Gamma(l+m+n+1)}}$$

Error function (or probability integral) :-

The error function or probability integral is defined as,

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt}$$

* This integral arises in the solutions of certain PDE and in the different applications of mathematics

Properties:- 1. $erf(-x) = -erf(x)$ 2. $erf(0) = 0$

3. $erf(\infty) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\infty e^{-t^2}\, dt = \dfrac{2}{\sqrt{\pi}}\cdot\dfrac{\sqrt{\pi}}{2} = 1$

Complementary error function, $erf_c(x)$, is defined as,

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\, dt$$

, and it follows that,

$$erf_c(x) = 1 - erf(x)$$

Problems and Solutions

01. Show that, $\Gamma(n) = \int_0^1 [\ln(1/y)]^{n-1} dy, \quad n>0$

Solution:

we have, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad (n>0)$

Put, $y = e^{-x}$

$\Rightarrow e^x = 1/y$

$\Rightarrow x = \ln(1/y)$

$\therefore \quad dx = y \cdot \left(-\frac{1}{y^2}\right) dy = -\frac{1}{y} dy$

$$\therefore \Gamma(n) = \int_1^0 [\ln(1/y)]^{n-1} \cdot y \cdot \left(-\frac{1}{y}\right) dy \quad ; n>0$$

$$\text{or, } \Gamma(n) = \int_0^1 [\ln(1/y)]^{n-1} dy \quad ; n>0$$

02. Express the following integrals in terms of Gamma functions:

a) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ b) $\int_0^{\pi/2} \tan^{1/2}\theta \, d\theta$ c) $\int_0^\infty a^{-bx^2} dx$

Solution: a) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

let, $x^2 = \sin\theta$

$\Rightarrow 2x\,dx = \cos\theta \, d\theta$

$\Rightarrow dx = \frac{1}{2} \sin^{-1/2}\theta \cos\theta \, d\theta$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\sin^{-1/2}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\sin^{-1/2}\theta \cos\theta \, d\theta}{\cos\theta}$$

$$= 1/2 \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta$$

$$= \frac{1}{4} \times 2 \int_0^{\pi/2} \sin^{(2\times 1/4 - 1)}\theta \cdot \cos^{(2\times 1/2 - 1)}\theta \, d\theta$$

$$= 1/4 \, \beta(1/4, 1/2) = \frac{1}{4} \cdot \frac{\Gamma(1/4) \cdot \Gamma(1/2)}{\Gamma(1/4 + 1/2)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

b) $\int_0^{\pi/2} \tan^{+1/2}\theta \, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2} \times 2 \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

$$= 1/2 \, \beta(3/4, 1/4) = \frac{1}{2} \cdot \frac{\Gamma(3/4) \cdot \Gamma(1/4)}{\Gamma(3/4 + 1/4)}$$

$$\text{or, } \int_0^{\pi/2} \tan^{1/2}\theta \, d\theta = 1/2 \, \Gamma(3/4) \cdot \Gamma(1/4)$$

c) $\int_0^\infty a^{-bx^2}\,dx = \int_0^\infty e^{-bx^2 \ln a}\,dx$

$\therefore x = \left[\frac{t}{(b\ln a)}\right]^{1/2}$

Let, $(b\ln a)x^2 = t$

$\therefore (b\ln a)\cdot 2x\,dx = dt$

$\Rightarrow dx = (b\ln a)^{-1}\cdot \frac{1}{2}\left[\frac{t}{b\ln a}\right]^{-1/2} dt$

$= \frac{1}{2\sqrt{b\ln a}}\cdot t^{-1/2}\,dt$

$$= \int_0^\infty e^{-t}\, t^{-1/2}\cdot \frac{1}{2\sqrt{b\ln a}}\,dt$$

$$= \frac{1}{2\sqrt{b\ln a}}\int_0^\infty e^{-t}\, t^{(1/2-1)}\,dt$$

$$\text{or, } \int_0^\infty a^{-bx^2}\,dx = \frac{1}{2\sqrt{b\ln a}}\,\Gamma(1/2) = \frac{\sqrt{\pi}}{2\sqrt{b\ln a}}$$

03. Show that, $\int_0^\infty x^{p-1} e^{-kx}\,dx \ (k>0) = \frac{\Gamma(p)}{k^p}$

Solution:- $\text{L.H.S} = \int_0^\infty x^{p-1} e^{-kx}\,dx \quad (k>0)$

Let, $kx = z$

$\Rightarrow k\,dx = dz$

$$= \int_0^\infty (z/k)^{p-1} e^{-z}\cdot \frac{dz}{k}$$

$$= \frac{1}{k^p}\int_0^\infty e^{-z} z^{p-1}\,dz = \frac{\Gamma(p)}{k^p} = \text{R.H.S (Proved)}$$

04. Show that, $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Solution:- $\text{L.H.S} = \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$

$$= 1/4 \times 2\int_0^{\pi/2} \sin^{1/2}\theta\,d\theta \times 2\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta$$

$$= 1/4 \times 2\int_0^{\pi/2} \sin^{(2\times 3/4-1)}\theta\,d\theta \cos^{(2\times 1/2-1)}\theta\,d\theta \times 2\int_0^{\pi/2} \sin^{(2\times 1/4-1)}\theta \cos^{(2\times 1/2-1)}\theta\,d\theta$$

$$= 1/4\ \beta(3/4, 1/2) \times \beta(1/4, 1/2)$$

$$= 1/4 \cdot \frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(3/4+1/2)} \cdot \frac{\Gamma(1/4)\cdot\Gamma(1/2)}{\Gamma(1/4+1/2)}$$

$$= \frac{1}{4}\cdot \frac{\Gamma(3/4)\cdot\sqrt{\pi}}{\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\cdot\sqrt{\pi}}{\Gamma(3/4)}$$

$$= \frac{\pi}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(1+1/4)} = \frac{\pi}{4}\cdot\frac{\Gamma(1/4)}{1/4\,\Gamma(1/4)} = \pi = \text{R.H.S (Proved)}$$

05. Evaluate the integral, $\int_0^1 x^3(1-\sqrt{x})^5\,dx$

Solution:-

$$\int_0^1 x^3(1-\sqrt{x})^5\,dx$$

Let, $x = z^2$

$\Rightarrow dx = 2z\,dz$

$$= \int_0^1 z^6(1-z)^5 \cdot 2z\,dz$$

$$= 2\int_0^1 z^7(1-z)^5\,dz$$

$$= 2\int_0^1 z^{(8-1)}(1-z)^{6-1}\,dz = 2\beta(8,6)$$

$$= 2\,\frac{\Gamma(8)\cdot\Gamma(6)}{\Gamma(14)} = \frac{2\cdot\cancel{\Gamma(8)}\cdot 5\cdot4\cdot3\cdot2\cdot1}{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot\cancel{\Gamma(8)}} = \frac{1}{13\cdot12\cdot11\cdot3} = \frac{1}{5148}$$

06. Show that, $\beta(m+1,n) + \beta(m,n+1) = \beta(m,n)$

Solution:-

$$\text{L.H.S} = \beta(m+1,n) + \beta(m,n+1)$$

$$= \frac{\Gamma(m+1)\cdot\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\cdot\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m)\cdot\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\cdot n\Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\cancel{(m+n)}\Gamma(m+n)}\cdot\cancel{(m+n)}$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$= \beta(m,n) = \text{R.H.S (Proved)}$$

07. Prove that, i) $\beta(m,\tfrac{1}{2}) = 2^{2m-1}\,\beta(m,m)$ } Legendre's duplication formula.

ii) $\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$

Solution:-

→ We know, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$

Putting, $n = \tfrac{1}{2}$ $\therefore \beta(m,\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta$

Now, again, putting, $n=m$

$$\therefore \beta(m,m) = 2\int_0^{\pi/2} (\sin\theta\cdot\cos\theta)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \cdot \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-1}}\; 2\int_0^{\pi/2} \sin^{(2m-1)}(2\theta)\, d\theta$$

$$= \frac{\cancel{2}\,1}{2^{(2m-1)}} \int_0^{\pi} \sin^{(2m-1)}\phi\, d\phi \qquad \left|\ \begin{array}{l} \text{Let, } 2\theta = \phi \\ \Rightarrow 2d\theta = d\phi \end{array}\right.$$

$$= \frac{1}{2^{(2m-1)}} \cdot 2\int_0^{\pi/2} \sin^{(2m-1)}\phi\, d\phi$$

$$= \frac{1}{2^{(2m-1)}} \cdot 2\int_0^{\pi/2} \sin^{(2m-1)}\phi\, \cos^{(2\times\frac{1}{2}-1)}\phi\, d\phi$$

$$\Rightarrow \beta(m,m) = \frac{1}{2^{(2m-1)}}\,\beta(m,\tfrac{1}{2})$$

$$\Rightarrow \boxed{\beta(m,\tfrac{1}{2}) = 2^{2m-1}\,\beta(m,m)}$$

$$\therefore \frac{\cancel{\Gamma(m)}\cdot\Gamma(\tfrac{1}{2})}{\Gamma(m+\tfrac{1}{2})} = 2^{2m-1} \cdot \frac{\cancel{\Gamma(m)}\cdot\Gamma(m)}{\Gamma(2m)}$$

$$\Rightarrow \Gamma(\tfrac{1}{2})\cdot\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\cdot\Gamma(m+\tfrac{1}{2})$$

$$\Rightarrow \boxed{\Gamma(m)\cdot\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

08. Prove that, $\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\cdot\frac{1\cdot3\cdot5\cdots\cdots(n-1)}{2\cdot4\cdot6\cdots\cdots n}$

Solution:- Let, $I = \int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}}$

putting, $x = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{\sin^n\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{1}{2}\times 2\int_0^{\pi/2} \sin^{\{2\times\left(\frac{n+1}{2}\right)-1\}}\theta \cdot \cos^{\{\frac{1}{2}\times2-1\}}\theta\, d\theta$$

$$\Rightarrow I = \frac{1}{2}\beta\left[\left(\frac{n+1}{2}\right), \frac{1}{2}\right] \text{———} (1)$$

Now, $\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left\{\left(\frac{n}{2}-\frac{1}{2}\right)+1\right\} = \Gamma\left\{\frac{(n-1)}{2}+1\right\}$

$$= \frac{(n-1)}{2}\cdot\frac{(n-3)}{2}\cdot\frac{(n-5)}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(1/2)$$

$$= \frac{1\cdot3\cdot5\cdots\cdots(n-1)}{2^n}\cdot\sqrt{\pi}$$

Again, $\Gamma(n/2+1) = n/2\,\Gamma(n/2)$

$$= \frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-3\right)\cdots 6\cdot4\cdot2\cdot\Gamma(1)$$

$$= \frac{2\cdot4\cdot6\cdots(n-2)\cdot n}{2^n}$$

From equation (1),

$$I = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\cdot\Gamma(1/2)}{\Gamma(n/2+1)} = \frac{1}{2}\cdot\frac{1\cdot3\cdot5\cdots(n-1)\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot4\cdot6\cdots n}$$

or, $\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\cdot\frac{1\cdot3\cdot5\cdots\cdots(n-1)}{2\cdot4\cdot6\cdots\cdots n}$ (Proved)