Electrostatic Potential

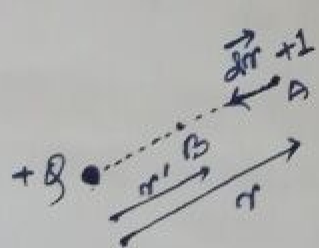
The electric lines of forces are not closed. The absence of closed lines is the property of vector field whose curl is zero. So the electric field is curl free ie irrotational. In vector notation

$$\vec{\nabla}\times\vec{E} = 0$$

Now since the curl of the gradient of a scalar quantity is zero, so the electric field can be expressed as the gradient of some scalar quantity, the electrostatic potential (V). Hence ~~...~~ due to this special property of the electric field the vector problems can be reduced to much simpler scalar problem.

Definition: Electrostatic potential at a point is defined as the workdone in bringing a unit test charge from infinity to that point.

Potential due to a point charge: Let us consider a point charge $+q$ due to which



an electric field is created surrounded by it.

Now force on the unit test charge exerted by q is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\times 1}{r^2}\hat{r}$$

The workdone to move the test charge by a very small distance dr is given by

$$dw = \vec{F}.\vec{dr}$$

$dW = -F\,dr$ [Since the displacement is occured in the opposite direction of force]

Now potential at a distance r is

$$V = -\int_{\infty}^{r} \frac{q}{4\pi\epsilon_0 r^2}\,dr$$

$$= \frac{q}{4\pi\epsilon_0 r}$$

Potential difference :- Potential difference between two points is the workdone to carry a unit test charge from one point to another.

If A and B are two points in the electric field which are at a distance r and r' respectively, then potential difference between A and B is

$$V_{AB} = -\int_{r}^{r'} \frac{q}{4\pi\epsilon_0 r^2}\,dr$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r'} - \frac{1}{r}\right)$$

$$= V_B - V_A$$

Potential energy :- In the electric field potential energy at a point means the workdone to carry a certain amount of charge from infinity to that point.

$\therefore$ Potential energy at a point = charge × potential at that point.

Relation between $\vec{E}$ and V :-

From the important property of $\vec{E}$

$$\vec{\nabla}\times\vec{E}=0$$

Then by Stoke's theorem, $\oint \vec{E}\cdot\vec{dl}=0$

So, the line integral is independent of path.

Now the potential at any point at a distance r from a source charge $+q$ is the work done to bring a unit test charge from infinity to that point.

So, $V(r) = -\int_{\infty}^{r}\vec{E}\cdot\vec{dr}$ [where $\vec{E}$ is force per unit charge at a distance r]

Let potential at a and b is $V(a)$ and $V(b)$ [$V(b) > V(a)$]

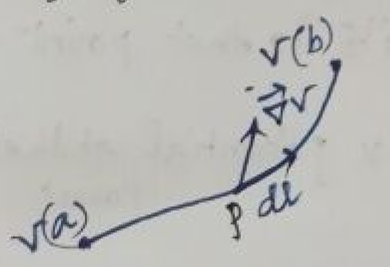
Then,

$$V(b)-V(a) = -\int_{\infty}^{b}\vec{E}\cdot\vec{dr} - \left[-\int_{\infty}^{a}\vec{E}\cdot\vec{dr}\right]$$

$$= -\int_{a}^{\infty}\vec{E}\cdot\vec{dr} - \int_{\infty}^{b}\vec{E}\cdot\vec{dr}$$

$$= -\int_{a}^{b}\vec{E}\cdot\vec{dr} \quad \text{.......(i)}$$

Let the unit test charge is carried out through a path as shown in the fig. below -



Here $\vec{\nabla}V$ denotes the rate of change of potential at any point P and remains constant for small elementary length dl, then rate of change of potential along $\vec{dl}$ is

$= \vec{\nabla}V \cdot d\vec{l}$

Now considering the whole path, the change in potential

$$V(b) - V(a) = \int_a^b (\vec{\nabla}V) \cdot d\vec{l} \quad \text{-------(ii)}$$

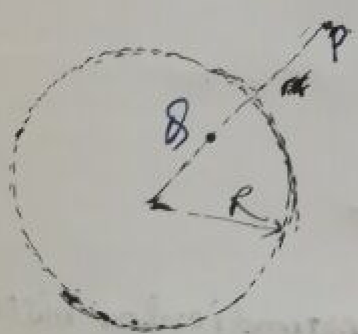
compairing eq^n (i) & (ii) we can say

$$\boxed{\vec{E} = -\vec{\nabla}V}$$

Unit of potential :- Since, the unit of electric field is N/C, so potential is measured in Nm/C or J/C. J/C is called Volt.

Note → It is not necessary that potential is zero at infinity. In that case where the potential is not zero at infinity then a referrence point is to be found out where the potential is zero.

Problem :- Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge.



Sol^n :- If q be the total charge, then electric field outside the sphere at a distance r $(r > R)$ is $\vec{E}_P = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$

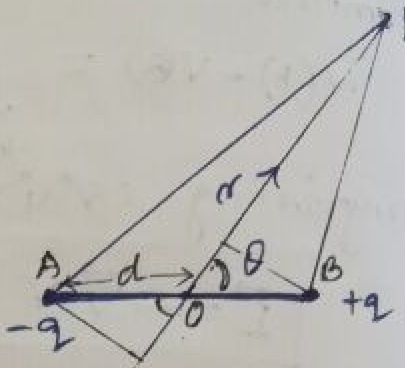
$$\text{So, } V_P = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r^2}\,dr = \frac{q}{4\pi\epsilon_0 r}$$

And potential inside the sphere at Q at a distance r (where $r < R$) is

$$V_Q = -\frac{1}{4\pi\epsilon_0}\int_\infty^R \frac{q}{r^2}\,dr - \int_R^r (0)\,dr \quad [\because \text{field inside the sphere is zero}]$$

$$= \frac{q}{4\pi\epsilon_0 R}$$

Potential at any point due to an electric dipole



Let two charges $+q$ and $-q$ separated by a distance $2d$ constitute an electric dipole having dipole moment $p = q \times 2d$. Due to this dipole potential at a distance r and at an angle θ is given by

$$V_P = \frac{q}{4\pi\epsilon_0 BP} + \frac{-q}{4\pi\epsilon_0 AP}$$

$$= \frac{q}{4\pi\epsilon_0 (r - d\cos\theta)} - \frac{q}{4\pi\epsilon_0 (r + d\cos\theta)}$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \frac{r + d\cos\theta - r + d\cos\theta}{(r - d\cos\theta)(r + d\cos\theta)} \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{2d\cos\theta}{(r^2 - d^2\cos^2\theta)}$$

$$= \frac{p\cos\theta}{4\pi\epsilon_0 (r^2 - d^2\cos^2\theta)}$$

For a point dipole $r \gg d$.

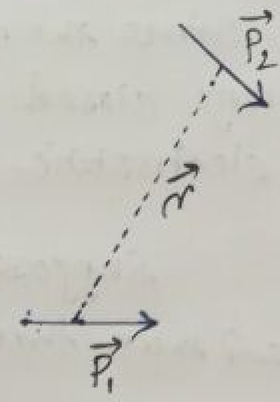
$$\therefore V_P = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

Special cases :- i) At $\theta = 90^\circ$ i.e. on the perpendicular bisector $V = 0$

ii) At $\theta = 0^\circ$, V is maximum ~~and is positive~~

iii) At $\theta = 180^\circ$, V is ~~maximum and is negative~~ minimum.

Mutual potential energy of two dipoles :



The mutual potential energy between two dipoles is equals to the work done in carring one dipole from infinity upto its position in the electric field produced by the other dipole. This can be easily determined by finding the electric field $\vec{E}_1$ due to the dipole $\vec{P}_1$ at the position of $\vec{P}_2$ and then calculating the potential energy of the dipole $\vec{P}_2$ placed in the electric field E_1 which is given by

$$U = -\vec{P}_2 \cdot \vec{E}_1 \;.$$

Now potential at the position of $\vec{P}_2$ due to P_1 is

$$V_1 = \frac{\vec{P}_1 \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{\vec{P}_1 \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

$$\therefore E_1 = -\vec{\nabla} V_1 = -\vec{\nabla}\left\{\frac{\vec{P}_1 \cdot \vec{r}}{4\pi\epsilon_0 r^3}\right\}$$

$$= -\frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r^3}\vec{\nabla}(\vec{P}_1 \cdot \vec{r}) + (\vec{P}_1 \cdot \vec{r})\vec{\nabla}\left(\frac{1}{r^3}\right)\right\}$$

$$= -\frac{1}{4\pi\epsilon_0}\left\{\frac{\vec{P}_1}{r^3} - \frac{3(\vec{P}_1 \cdot \vec{r})\vec{r}}{r^5}\right\}$$

$$\left[\because \vec{\nabla}\left(\frac{1}{r^3}\right) = -\frac{3\vec{r}}{r^5}\right]$$

So, the mutual potential energy of the two dipole is given by

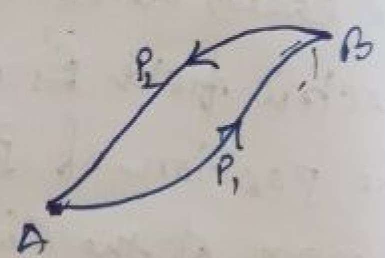
$$U = -\vec{P}_2 \cdot \vec{E}_1 = \frac{1}{4\pi\epsilon_0}\left\{\frac{\vec{P}_1 \cdot \vec{P}_2}{r^3} - \frac{3(\vec{P}_1 \cdot \vec{r})(\vec{P}_2 \cdot \vec{r})}{r^5}\right\}$$

Conservative nature of electrostatic field :-

Since the workdone to carry a unit test charge around any closed path in a static field is zero, so the electrostatic field is conservative force field.

Suppose a unit test chare is moved in a static field through AP_1B and then through BP_2A e.g. it reaches to its initial point.

Now the workdone to move the unit test charge from A to B through AP_1B is



$$w_{AB} = -\int_{AP_1B} \vec{E}\cdot\vec{dl}$$

Similarly workdone through BP_2A is

$$w_{BA} = -\int_{BP_2A} \vec{E}\cdot\vec{dl}$$

So, total workdone $= -\int_{AP_1B} E\cdot dl - \int_{BP_2A} \vec{E}\cdot\vec{dl}$

$$= -\oint \vec{E}\cdot\vec{dl}$$

$$= 0$$

So, the workdone is independent of path and depends only the initial and final points.