CLASSICAL DYNAMICS

Paper:DSE-IV Semester:VI

Calculus of variation

Part-II

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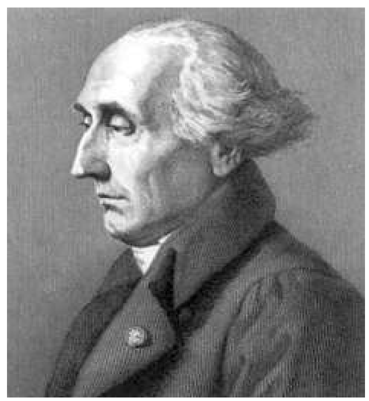
Hamilton's Variational Principle

Jean le Rond d'Alembert (1717 – 1783)

We have already derived **Lagrange's equation** in Semester III considering the instantaneous state of the system and small virtual displacements about the instantaneous state, i.e., using a Differential Principle (**a LOCAL principle**) such as **D'Alembert 's Principle,** which is essentially based on **Newton's Law**.



Joseph Louis Lagrange (1736 – 1813)

Sir William Rowan Hamilton (1805 – 1865)

In Semester VI, we shall derive **Lagrange's equation** from **Hamilton's Principle:** An Integral (or Variational) Principle (a **GLOBAL** principle) that considers the entire motion of the system between times t_1 and t_2, and small virtual variations of this motion from the actual motion.

Hamilton's Variational Principle

Since the state of a particle is specified by its ***position*** and ***velocity*** at a particular ***time***, we look for some function of those variables to work with. Then we look for a general principle involving this function that tells us how the external world influences the particle's state.

For a system of N particles, the system may be considered as
a ***single particle*** moving along a trajectory in a $3N$ dimensional space.
The space is referred as ***configuration space*** and the single particle as ***system point***.

There is ***no necessary connection*** between ***configuration space*** (expressed in generalised coordinates) and ***physical space*** (expressed in standard position coordinates). The generalised coordinates need not even be position coordinates. Thus the trajectory of the system point in configuration space need not have any necessary resemblance to the path in space of any actual particle. **Each point on the path represents the entire system configuration at some given instant of time.**

Hamilton's Variational Principle

Of all possible paths along which a dynamical system may move from one point to another within a given time interval (consistent with constraints, if any), the actual path followed by the system is the one for which the line integral of Lagrangian is extremum .

Thus, the motion of a dynamical system from t_0 *to* t_1 is such that the line integral $\mathbf{I} = \int L\, \mathbf{dt}$ is extremum for actual path. Mathematically,

$$\delta \int_{t_0}^{t_1} L dt = 0$$

the action or action integral

where $L = L(q_j, \dot{q}_j, t)$ is the Lagrangian of the system.

Alternatively

Hence we can also define the Hamilton's principle as "Out of all possible paths of a dynamical system between the time instants t_0 and t_1, the actual path followed by the system is one for which the action has a stationary value"

In physics, **action** is an attribute of the dynamics of a physical system from which the equations of motion of the system can be derived through the principle of stationary action. Action has the dimensions of [energy]·[time] or [momentum]·[length], and its SI unit is joule-second.

Derivation of Lagrange's equation from Hamilton's Principle

$$I = \int_{t_0}^{t_1} L dt \qquad \ldots (1)$$

where L is the Lagrangian of the system.

$$\delta I = \delta \int_{t_0}^{t_1} L dt,$$

$$= \int_{t_0}^{t_1} \left[\sum_j \frac{\partial L}{\partial q_j} \delta q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right] dt$$

As there is no variation in time along any path, hence $\delta t = 0$.

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} \delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j dt.$$

Since $\qquad \delta \dfrac{dq_j}{dt} = \dfrac{d}{dt}\left(\delta q_j\right),$

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j} \delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt}\left(\delta q_j\right) dt \qquad \ldots (2)$$

Lagrange's Equation from Hamilton's Principle

Integrating the second integral on the r. h. s. of equation (2) we get

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1}\sum_j \frac{\partial L}{\partial q_j}\delta q_j dt + \left[\sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1}\sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j dt.$$

Since there is no variation in the co-ordinates along any path at the end points, hence change in the co-ordinates at the end points is zero. i.e., $\left(\delta q_j\right)_{t_0}^{t_1} = 0$.

Thus we have

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1}\sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j dt. \qquad ...(3)$$

If the system is holonomic, then all the generalized co-ordinates are linearly independent and hence we have

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \int_{t_0}^{t_1}\sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j dt = 0$$

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0. \qquad ...(4)$$

Which are the required Lagrange's equation of motion.

Applications of Hamilton's Variational Principle

Example 1 : Use Hamilton's principle to find the equations of motion of a particle of unit mass moving on a plane in a conservative force field.

Solution: Let the force $\bar{F}$ be conservative and under the action of which the particle of unit mass be moving on the xy plane. Let P (x, y) be the position of the particle.

We write the force

$$\bar{F} = iF_x + jF_y .$$

Since $\bar{F}$ is conservative, we have therefore,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y} .$$

The kinetic energy of the particle is given by

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right).$$

Hence the Lagrangian of the particle becomes

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x, y). \qquad \ldots (1)$$

Applications of Hamilton's Variational Principle

The Hamilton's principle states that

$$\delta \int_{t_0}^{t_1} L dt = 0, \qquad \ldots (2)$$

$$\int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} \right] dt = 0,$$

$$\Rightarrow \qquad \int_{t_0}^{t_1} \left[\left(\dot{x} \delta \dot{x} + \dot{y} \delta \dot{y} \right) - \frac{\partial V}{\partial x} \delta x - \frac{\partial V}{\partial y} \delta y \right] dt = 0. \qquad \ldots (3)$$

Consider

$$\int_{t_0}^{t_1} \dot{x} \delta \dot{x} dt = \int_{t_0}^{t_1} \dot{x} \frac{d}{dt} (\delta x) dt$$

Integrating by parts we get

$$\int_{t_0}^{t_1} \dot{x} \delta \dot{x} dt = \left(\dot{x} \delta x \right)_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x} \left(\delta x \right) dt .$$

Since $\delta x = 0$ at both the ends t_0 and t_1 along any path, therefore,

$$\int_{t_0}^{t_1} \dot{x} \delta \dot{x} dt = - \int_{t_0}^{t_1} \ddot{x} \left(\delta x \right) dt . \qquad \ldots (4)$$

Applications of Hamilton's Variational Principle

Similarly, we have

$$\int_{t_0}^{t_1} \dot{y}\delta\dot{y}dt = -\int_{t_0}^{t_1} \ddot{y}(\delta y)dt . \qquad \ldots (5)$$

Substituting these values in equation (3) we get

$$\int_{t_0}^{t_1}\left[\left(\ddot{x}+\frac{\partial V}{\partial x}\right)\delta x+\left(\ddot{y}+\frac{\partial V}{\partial y}\right)\delta y\right]dt=0 .$$

Since δx and δy are independent and arbitrary, then we have

$$\ddot{x}+\frac{\partial V}{\partial x}=0, \qquad \ddot{y}+\frac{\partial V}{\partial y}=0 .$$

$$\begin{aligned} \ddot{x} &= -\frac{\partial V}{\partial x}=F_x, \\ \ddot{y} &= -\frac{\partial V}{\partial y}=F_y. \end{aligned} \qquad \ldots (6)$$

These are the equations of motion of a particle of unit mass moving under the action of the conservative force field.

Applications of Hamilton's Variational Principle

Example 2: Use Hamilton's principle to find the equation of motion of a simple pendulum.

Solution: In case of a simple pendulum, the only generalized co-ordinate is θ, and the Lagrangian is given by

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right). \qquad \ldots(1)$$

The Hamilton's Principle states that "the path followed by the pendulum is one along which the line integral of Lagrangian is extremum". i.e.,

$$\delta\int_{t_0}^{t_1} \mathrm{L}dt = 0,$$

$$\int_{t_0}^{t_1} \delta\left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right)\right]dt = 0,$$

$$\int_{t_0}^{t_1} \left[ml^2\dot{\theta}\delta\dot{\theta} - mgl\sin\theta\delta\theta\right]dt = 0.$$

Since, we have $\delta\dfrac{d}{dt} = \dfrac{d}{dt}\delta$.

Therefore, $$\int_{t_0}^{t_1}\left[ml^2\dot{\theta}\frac{d}{dt}\left(\delta\theta\right) - mgl\sin\theta\delta\theta\right]dt = 0.$$

Applications of Hamilton's Variational Principle

Integrating the first integral by parts we get

$$ml^2\left(\dot{\theta}\delta\theta\right)_{t_0}^{t_1} - \int_{t_0}^{t_1} m\left[l^2\ddot{\theta} + gl\sin\theta\right]\delta\theta dt = 0.$$

Since $\left(\delta\theta\right)_{t_0}^{t_1} = 0$, we have therefore,

$$\int_{t_0}^{t_1} m\left[l^2\ddot{\theta} + gl\sin\theta\right]\delta\theta dt = 0.$$

As $\delta\theta$ is arbitrary, we have

$$l^2\ddot{\theta} + gl\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This is the required equation of motion of the simple pendulum.

Applications of Hamilton's Variational Principle

Example 3. Apply variational principle to find the equation of motion of one dimensional harmonic oscillator.

Solution: : The Lagrangian L for one dimensional harmonic oscillator is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \text{ or } L = f(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

According to Hamilton's principle or variational principle $\int L\,dt$ or $\int f(x, \dot{x}, t)\,dt$ is extremum.

Euler-Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \frac{\partial f}{\partial x} = 0$$

Here $\quad \frac{\partial f}{\partial x} = -kx$, $\frac{\partial f}{\partial \dot{x}} = m\dot{x}$

Therefore, $\quad m\ddot{x} + kx = 0$

which is the equation of motion for one-dimensional harmonic oscillator.

Example 4: *A particle of mass m falls a given distance z_0 in time $t_0 = \sqrt{2z_0/g}$ and the distance travelled in time t is given by $z = at + bt^2$, where constants a and b are such that the time t_0 is always the same. Show that the integration $\int_0^{t_0} L\,dt$ is an extremum for real values of the coefficients only when $a = 0$ and $b = g/2$.*

Solution: Let the particle fall from O $(z = 0)$ to P $(OP = z)$ in time t.

Kinetic energy of the particle at P,

$$T = \frac{1}{2} m\dot{z}^2 .$$

Potential energy of the particle at P, $V = -mgz$

Hence $$L = T - V = \frac{1}{2} m\dot{z}^2 + mgz \quad (1)$$

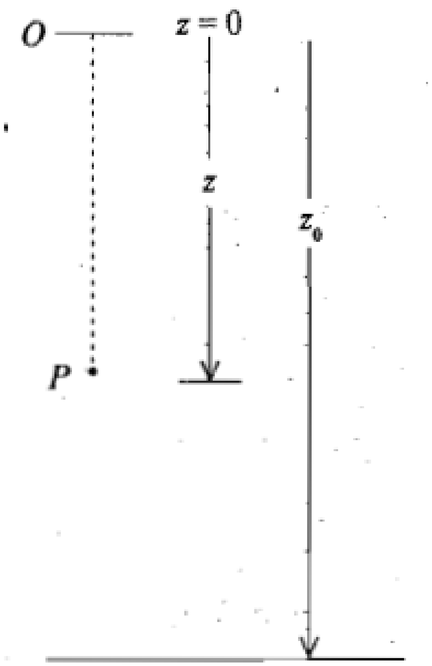
According to the Hamilton's principle

$$\delta \int_0^{t_0} L\,dt = 0$$

or $$\int_0^{t_0} L\,dt = \text{extremum, for which}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] - \frac{\partial L}{\partial z} = 0 \quad (2)$$

is to be satisfied.



Here, $\frac{\partial L}{\partial \dot{z}} = m\dot{z}$ and $\frac{\partial L}{\partial z} = mg$. Hence (2) becomes

$$\frac{d}{dt}(m\dot{z}) - mg = 0 \text{ or } \ddot{z} = g \qquad (3)$$

But $z = at + bt^2$ and therefore $\dot{z} = a + 2bt$ and $\ddot{z} = 2b$ (4)

From (3) and (4) we get

$$2b = g \text{ or } b = g/2 \qquad (5)$$

Also at $t = t_0$, $z = z_0$, we have

$$z_0 = at_0 + bt_0^2 \qquad (6)$$

But
$$t_0 = \sqrt{\frac{2z_0}{g}} \text{ or } z_0 = \frac{1}{2}gt_0^2 \qquad (7)$$

Comparing (6) and (7) and putting $b = g/2$, we get

$$at_0 + \frac{g}{2}t_0^2 = \frac{1}{2}gt_0^2 \text{ or } at_0 = 0$$

Since $t_0 \neq 0$, therefore, $a = 0$.

Thus we find that $\int_0^{t_0} L\,dt$ is extremum, when $a = 0$, $b = g/2$.

Applications of Hamilton's Variational Principle

Example 5: Imagine that we have a particle than can move in one dimension (i.e. one coordinate – for example its height y above a table – suffices to describe its position). Using Hamilton's Principle find the equation of motion.

Solution: We are going to use the variational principle to find the equation of motion

Its kinetic energy is $T = \frac{1}{2}m\dot{y}^2$ and potential energy is $V = mgy$.

The lagrangian of the system is $L = \frac{1}{2}m\dot{y}^2 - mgy$

According to Hamilton's Principle, $\delta\int_{t_1}^{t_2} L\,dt = 0$

$$m\delta\int_{t_1}^{t_2}\left(\tfrac{1}{2}\dot{y}^2 - gy\right)dt = 0.$$

$$m\int_{t_1}^{t_2}(\dot{y}\,\delta\dot{y} - g\,\delta y)dt = 0$$

$$I_1 - I_2 = 0, \text{ say}$$

Therefore $I_1 = m\int_{t_1}^{t_2}\dot{y}d\delta y$

By integration by parts: $I_1 = [m\dot{y}\,\delta y]_{t_1}^{t_2} - m\int_{t_1}^{t_2}\delta y\,d\dot{y}$

The first term is zero because the variation is zero at the beginning and end points. In the second term, $d\dot{y} = \ddot{y}\,dt$, and therefore

Applications of Hamilton's Variational Principle

$$I_1 = -m\int_{t_1}^{t_2} \ddot{y}\,\delta y\,dt$$

$$\therefore \qquad \delta\int_{t_1}^{t_2} L\,dt = -m\int_{t_1}^{t_2} (\ddot{y} + g)\delta y\,dt$$

and, for this to be zero, we must have

$$\ddot{y} = -g\,.$$

Hamiltonian formalism

Lagrange formulation is in terms of generalized coordinates *q* and generalized Velocities gives the $\dot{q}$uations of motion, which are second order in time. Instead if we regard N generalized coordinates *q* and N generalized momenta *p* as independent variables, and again *q* and *p* are defined at every instant of time t, we will get 2N 1st order equations. Hence the 2N equations of motion describe the behaviour of the system in a phase space whose coodinates are the 2N independent variables. These are called canonical coordinates and canonical momenta. This new formulation is known as Hamiltonian formulation.

Hamilton's equation from Hamilton's Principle

We know the action of a particle is defined by

$$I = \int_{t_0}^{t_1} L dt \qquad \ldots (1)$$

where L is the Lagrangian of the system. If $H\left(p_j, q_j, t\right)$ is the Hamiltonian of the motion then we have by definition

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (2)$$

Replacing L in equation (1) by using (2) we have the action in mechanics as

$$I = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt. \qquad \ldots (3)$$

Now by Hamilton's principle, we have

$$\delta \int_{t_0}^{t_1} L dt = 0 \quad \Rightarrow \quad \delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt = 0. \qquad \ldots (4)$$

This is known as the modified Hamilton's principle. Thus we have

$$\delta \int_{t_0}^{t_1} L dt = \quad \delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt,$$

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j \delta p_j \dot{q}_j + \sum_j p_j \delta \dot{q}_j - \sum_j \frac{\partial H}{\partial q_j} \delta q_j - \sum_j \frac{\partial H}{\partial p_j} \delta p_j - \frac{\partial H}{\partial t} \delta t \right] dt.$$

Hamilton's equation from Hamilton's Principle

Since time is fixed along any path, hence change in time along any path is zero. i.e.,

$\delta t = 0$ along any path. Hence above equation becomes

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1}\left[\sum_j\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j + \sum_j p_j\delta\dot{q}_j - \sum_j \frac{\partial H}{\partial q_j}\delta q_j\right]dt \qquad \ldots (5)$$

Now consider the integral

$$\int_{t_0}^{t_1}\sum_j p_j\delta\dot{q}_j dt = \int_{t_0}^{t_1}\sum_j p_j\frac{d}{dt}(\delta q_j)\, dt\,.$$

Integrating the integral on the r. h. s. by parts we get

$$\int_{t_0}^{t_1}\sum_j p_j\delta\dot{q}_j dt = \left(\sum_j p_j\delta q_j\right)_{t_0}^{t_1} - \int_{t_0}^{t_1}\sum_j \dot{p}_j\delta q_j dt\,.$$

Since $\left(\delta q_j\right)_{t_0}^{t_1} = 0$. We have therefore

$$\int_{t_0}^{t_1}\sum_j p_j\delta\dot{q}_j dt = -\int_{t_0}^{t_1}\sum_j \dot{p}_j\delta q_j dt\,.$$

Substituting this in equation (5) we get

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1}\left[\sum_j\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j + \sum_j\left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta q_j\right]dt\,.$$

Hamilton's equation from Hamilton's Principle

Now we see that

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \int_{t_0}^{t_1}\left[\sum_j\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j + \sum_j\left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta q_j\right]dt = 0\,.$$

For holonomic system we have q_j, p_j are independent, hence

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \dot{q}_j - \frac{\partial H}{\partial p_j} = 0, \quad \dot{p}_j + \frac{\partial H}{\partial q_j} = 0\,.$$

$$\Rightarrow \quad \delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \qquad \ldots (6)$$

These are the Hamilton's canonical equations of motion.

Hamilton's canonical equations of motion are the necessary and sufficient conditions for the action to have stationary value.

Lagrangian from Hamiltonian and conversely..

The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (1)$$

which satisfies the Hamilton's canonical equations of motion.

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad . \qquad \ldots (2)$$

Now from equation (1) we find the Lagrangian

$$L = \sum_j p_j \dot{q}_j - H \ . \qquad \ldots (3)$$

Thus, $\dfrac{\partial L}{\partial q_j} = -\dfrac{\partial H}{\partial q_j}, \quad and \quad \dfrac{\partial L}{\partial \dot{q}_j} = p_j.$

Hence,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{\partial H}{\partial q_j} - \frac{d}{dt}\left(p_j\right),$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j - \dot{p}_j,$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0.$$

This shows that the equation (3) gives the required Lagrangian which satisfies the Lagrange's equations of motion.

Lagrangian from Hamiltonian and conversely..

The Hamiltonian H in terms of Lagrangian L is defined as

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (1)$$

where L satisfies the Lagrange's equations of motion viz.,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0, \qquad \ldots (2)$$

$$\Rightarrow \qquad \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right),$$

$$= \frac{d}{dt}(p_j).$$

$$\Rightarrow \qquad \frac{\partial L}{\partial q_j} = \dot{p}_j. \qquad \ldots (3)$$

Now from equation (1) we find

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j}. \qquad \ldots (4)$$

Lagrangian from Hamiltonian and conversely..

From equations (3) and (4) we have

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j . \qquad \ldots (5)$$

Similarly, we find from equation (1)

$$\frac{\partial H}{\partial p_j} = \dot{q}_j . \qquad \ldots (6)$$

Equations (5) and (6) are the required Hamilton's equations of motion.

Applications of Hamilton's equation

Example1: Describe the motion of a particle of mass m moving near the surface of the Earth under the Earth's constant gravitational field by Hamilton's procedure.

Solution: Consider a particle of mass m moving near the surface of the Earth under the Earth's constant gravitational field. Let (x, y, z) be the Cartesian co-ordinates of the projectile, z being vertical. Then the Lagrangian of the projectile is given by

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - mgz. \qquad \ldots (1)$$

We see that the generalized co-ordinates x and y are absent in the Lagrangian, hence they are the cyclic co-ordinates. This implies that any change in these co-ordinates can not affect the Lagrangian. This implies that the corresponding generalized momentum is conserved. In this case the generalized momentum is the linear momentum and is conserved.

i.e.,
$$\begin{aligned} p_x &= m\dot{x} = const. \\ p_y &= m\dot{y} = const. \\ p_z &= m\dot{z}. \end{aligned} \qquad \ldots (2)$$

This shows that the horizontal components of momentum are conserved.

The Hamiltonian of the particle is defined by

$$H = \sum_j p_j \dot{q}_j - L,$$

Applications of Hamilton's equation

$$H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + mgz . \qquad \ldots (3)$$

Eliminating $\dot{x}, \dot{y}, \dot{z}$ between equations (2) and (3) we get

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz . \qquad \ldots (4)$$

The Hamilton's equations of motion give

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0,\ \dot{p}_y = -\frac{\partial H}{\partial y} = 0,\ \dot{p}_z = -\frac{\partial H}{\partial z} = -mg . \qquad \ldots (5)$$

and

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m},\ \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m},\ \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} . \qquad \ldots (6)$$

From these set of equations we obtain

$$\ddot{x} = 0,\ \ddot{y} = 0,\ \ddot{z} = -g \qquad \ldots (7)$$

These are the required equations of motion of the projectile near the surface of the Earth.

Applications of Hamilton's equation

Example 2: Obtain the Hamiltonian H and the Hamilton's equations of motion of a simple pendulum. Prove or disprove that H represents the constant of motion and total energy.

Solution: The Example is solved earlier by various methods. The Lagrangian of the pendulum is given by

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta), \qquad \ldots (1)$$

where the generalized momentum is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{m l^2}. \qquad \ldots (2)$$

The Hamiltonian of the system is given by

$$H = p_\theta \dot{\theta} - L,$$

$$\Rightarrow \quad H = p_\theta \dot{\theta} - \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta).$$

Eliminating $\dot{\theta}$ we obtain

$$H = \frac{p_\theta^2}{2 m l^2} + mgl(1 - \cos\theta). \qquad \ldots (3)$$

Applications of Hamilton's equation

Hamilton's canonical equations of motion are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

These equations give

$$\dot{\theta} = \frac{p_\theta}{ml^2}, \quad \dot{p}_\theta = -mgl\sin\theta. \qquad \ldots(4)$$

Now eliminating p_θ from these equations we get

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \qquad \ldots(5)$$

This is the equation of motion of the system.

Now, $$\frac{dH}{dt} = \frac{p_\theta \dot{p}_\theta}{ml^2} + mgl\sin\theta\,\dot{\theta},$$

$$= ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta},$$

$$= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right),$$

Hence, $\frac{dH}{dt} = 0$. This proves that H is a constant of motion.

Applications of Hamilton's equation

Now to see whether H represents total energy or not, we consider

$$T+V=\frac{1}{2}ml^2\dot{\theta}^2+mgl\left(1-\cos\theta\right).$$

Using equation (4) we eliminate $\dot{\theta}$ from the above equation, we obtain

$$T+V=\frac{p_\theta^2}{2ml^2}+mgl\left(1-\cos\theta\right). \qquad \ldots(6)$$

This is as same as the Hamiltonian H from equation (3). Thus Hamiltonian H represents the total energy of the pendulum.

Example 3: Find the equations of motion of a particle in a central field of force using Hamilton's equations.

Solution: In such a field the force is always directed to the centre.

$$F = -\frac{K}{r^2} = -\frac{\partial V}{\partial r};\ V(r) = -\frac{K}{r} \tag{1}$$

If m be the mass of a particle moving in the central force field, then the Lagrangian L in polar coordinates can be expressed as

$$L = T - V(r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{2}$$

In order to write the Hamiltonian, $\dot{r}$ and $\dot{\theta}$ must be replaced by the generalized momenta p_r and p_θ. Now

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

or

$$\dot{r} = \frac{p_r}{m} \text{ and } \dot{\theta} = \frac{p_\theta}{mr^2} \tag{3}$$

Hence

$$H = T + V = \frac{1}{2}m\left[\left[\frac{p_r}{m}\right]^2 + r^2\left[\frac{p_\theta}{mr^2}\right]^2\right] + V(r)$$

or

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + V(r) \tag{4}$$

Applications of Hamilton's equation

The Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } -\dot{p}_k = \frac{\partial H}{\partial q_k}.$$

Here $k = r$ and θ ; hence in the present case, the equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{5}$$

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \tag{6}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{7}$$

and

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \tag{8}$$

From eqs. (7) and (8)), we get

$$p_\theta = \text{constant} = mr^2\dot{\theta} \tag{9}$$

Applications of Hamilton's equation

This is the *familiar equation of conservation of angular momentum of a particle, moving in a central force field*. From eqs. (5) and (6) we have

$$-m\ddot{r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \text{ or } m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = 0 \qquad (10)$$

This is an important differentail equation in second order for a particle moving under central force. In case of square law force, $V(r) = -K/r$ and $f(r) = -\partial V/\partial r = -K/r^2$. Then

$$m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{K}{r^2} = 0 \qquad (11)$$

Symmetries and conservation laws

One very important discovery has been the link between conservation laws and basic symmetries in nature. For example, empty space possesses the symmetries that it is the same at every location (homogeneity) and in every direction (isotropy); these symmetries in turn lead to the invariance principles that the laws of physics should be the same regardless of changes of position or of orientation in space. The first invariance principle implies the law of conservation of linear momentum, while the second implies conservation of angular momentum. The symmetry known as the homogeneity of time leads to the invariance principle that the laws of physics remain the same at all times, which in turn implies the law of conservation of energy.

Recall:

With the generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$, the EL equations take the form

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

Thus, if L does not depend on q_i (i.e. q_i is cyclic), then the conjugate momentum is conserved over time, i.e. it is a constant of motion.

Symmetries and conservation laws

Homogeneity of Time and Energy Conservation

If you calculate the equations of motion for a Lagrangian starting for time t_0 to t_1, does the Lagrangian and hence the equations of motion change if we calculate the equations of motion for the same time span but starting at a later time? That is, if we shift time by Δt and make the transformation $t \rightarrow t' = t + \Delta t$ does the Lagrangian change? To answer this question we analyze the total time derivative of the Lagrangian.

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

If the Lagrange function would be time dependent we would get an additional term with the partial derivative of the Lagrangian with respect to time $\frac{\partial L}{\partial t}$. By use of the Euler-Lagrange Equation, we replace $\frac{\partial L}{\partial q_i}$ by $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$.

$$\frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

Now we see, that this is nothing but the derivative of the product

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

Rearranging terms we have

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

Thus whatever is in the Brackets is conserved.

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = const. \tag{1}$$

Symmetries and conservation laws

We define this constant as the **Hamiltionian** H. Since there is no explicit time dependence of the Lagrangian, the Lagrangian is invariant with respect to time transformations $t \to t' = t + \Delta t$. So far so good, but what does the conserved quantity represent? For a Lagrange function of the form $L = T(\dot{q}_i) - V(q_i)$ it is true that

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

now plugging this into, equation (1)

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

becomes

$$H = 2T - L = 2T - T + V = T + V = E$$

Time Symmetry: A time homogeneous (independent) Lagrangian is invariant under time translation. The energy of the system is conserved and equals the Hamiltonian.

Symmetries and conservation laws

Isotropy of Space and Angular Momentum Conservation

Isotropy means the property that rotational translation does not alter the Lagrangian. An infinitesimal rotation $\delta\vec{\varphi}$ leaves the Lagrangian invariant

For a particle with position vector $\mathbf{r}$ from the origin. The change in $\delta\mathbf{r}$ due to the infinitesimal rotation is proportional to the distance $\mathbf{r}$.

$$\delta\mathbf{r} = \delta\vec{\varphi} \times \mathbf{r}$$

the time derivative of this relation gives a similar expression for velocity

$$\delta\mathbf{v} = \delta\vec{\varphi} \times \mathbf{v}$$

The change in the Lagrangian due to rotation is

$$\delta L = \sum_i \left(\frac{\partial L}{\partial r_i} \delta r_i + \frac{\partial L}{\partial v_i} \delta v_i \right) = 0.$$

These equations together with the definitions

$$\frac{\partial L}{\partial r_i} = \dot{p}_i$$

$$\frac{\partial L}{\partial v_i} = p_i$$

give

$$\delta L = (\dot{\mathbf{p}} \cdot \delta\vec{\varphi} \times \mathbf{r} + \mathbf{p} \cdot \delta\vec{\varphi} \times \mathbf{v}) = 0$$

Symmetries and conservation laws

we use that even permutations of $\dot{\mathbf{p}} \cdot \delta\vec{\varphi} \times \mathbf{r}$ are the same $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$,

$$\delta L = \delta\vec{\varphi}\,(\mathbf{r} \times \dot{\mathbf{p}} + \mathbf{p} \times \mathbf{v}) = 0$$

$$\delta L = \delta\vec{\varphi}\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = 0$$

$\delta\vec{\varphi}$ is arbitrary, thus for the Lagrangian to be invariant we need

$$\frac{d}{dt}\sum_i \mathbf{r} \times \mathbf{p} = 0$$

to be zero.

Which means the **angular momentum is conserved**.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = const.$$

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