CLASSICAL DYNAMICS

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Calculus of variation

Part-I

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Calculus of variation

Solution of a dynamical problem means that we want to locate the position of the system at a particular instant of time. Sometimes we are also interested on the path followed by the system.

The piece wise information of the path $y=f\,(x)$, whether it is minimum or maximum at a point can be obtained from differential calculus by putting $y'=0$. The function is either maximum or minimum at a point $x=a$ depends upon the value of second derivative of the function y'' at that point. The function is maximum at $x=a$ if $y''(a)<0$ and is minimum if $y''(a)>0$.

However, if we want to know the information about the whole path, we use integral calculus. i.e., the techniques of calculus of variation (variational principle). Thus the calculus of variation has its origin in the generalization of the elementary theory of maxima and minima of function of a single variable or more variables. The history of calculus of variations can be traced back to the year 1696, when John Bernoulli advanced the problem of the brachistochrone.

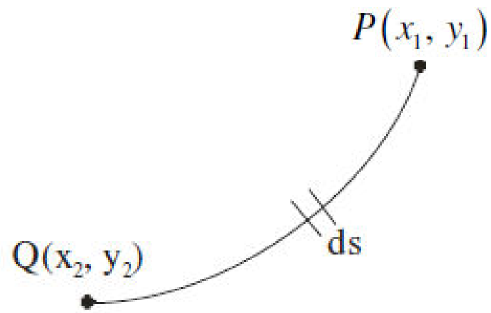
Calculus of variation

Consider the motion of a particle or system of particles along a curve $y = f(x)$ joining two points $P(x_1,y_1)$ *and* $Q(x_2, y_2)$. The infinitesimal distance between two points on the curve is given by

$$ds = \left(dx^2 + dy^2\right)^{\frac{1}{2}}$$

Hence the total distance between two point P and Q along the curve is given by

$$I\left(y(x)\right) = \int_{x_1}^{x_2} \left(1 + y'^2\right)^{\frac{1}{2}} dx, \quad y' = \frac{dy}{dx}$$



$$ds = \left(dx^2 + dy^2\right)^{\frac{1}{2}}$$
$$= \left(1 + y'^2\right)^{\frac{1}{2}} dx$$

In general the integrand is a function of the independent variable *x*, the dependent variable *y* and its derivative *y'*. Thus the most general form of the integral is given by

$$I = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx.$$

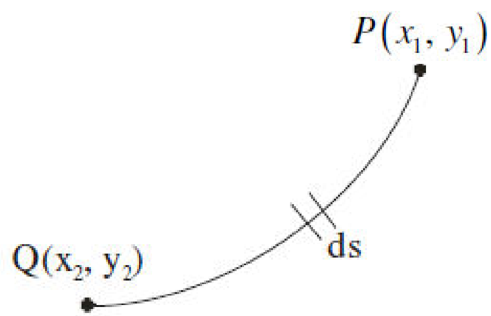
Calculus of variation

Formulation of the problem of calculus of variation

The integral $I = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx.$ may represent ***the total path between two given points, the surface area of revolution of a curve, the time for quickest decent etc.*** depending upon the situation of the problem.

The functional *I*, in general, depends upon the starting point P(x_1,y_1), the end point Q(x_2 ,y_2) and the curve between two points.

The question is what condition is to be satisfied by $y(x)$ such that the functional $I(y(x))$ must have an extremum value.



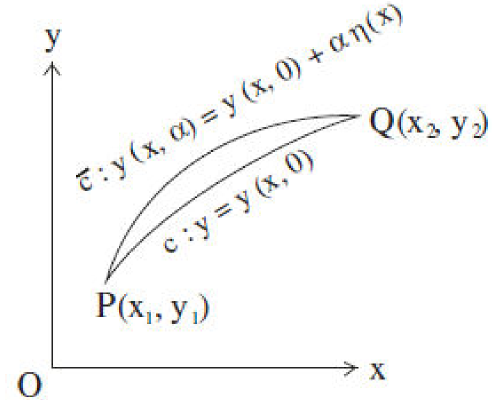
$$ds = \left(dx^2 + dy^2\right)^{\frac{1}{2}}$$
$$= \left(1 + y'^2\right)^{\frac{1}{2}} dx$$

The calculus of variations is concerned with the maxima or minima (collectively called **extrema**) of functionals. A functional is described as "functions of functions" which means a quantity whose values are determined by one or several functions.

Calculus of variation

Our aim is to find a curve (path) between P and Q for which the integral

$$I\left(y(x)\right)=\int_{x_1}^{x_2} f\left(x,\, y,\, y'\right)dx \qquad \ldots (1)$$



is an extremum. So, we have to take into account all possible paths between these two points . To include other paths, we require another parameter , say α.

We can label all possible paths starting from P and ending at Q by the family of equations

$$y(x,\alpha)=y(x,0)+\alpha\eta(x), \qquad \ldots (2)$$

where α is a parameter and $\eta(x)$ is any differentiable function of x.

Calculus of variation

For different values of α we get different curves. Accordingly the value of the integral I will be different for different paths. Since y is prescribed at the end points, this implies that there is no variation in y at the end points. i.e., all the curves of the family must be identical at fixed points P and Q.

$$\Rightarrow \quad \eta(x_1) = 0 = \eta(x_2) \qquad \ldots (3)$$

Conversely, the condition (3) ensures us that the curves of the family that all pass through the points P and Q. Let the value of the functional along the neighboring curve be given by

$$I(y(x,\alpha)) = \int_{x_1}^{x_2} f(x, y(x,\alpha), y'(x,\alpha))\,dx \qquad \ldots (4)$$

Calculus of variation

From differential calculus, we know the integral I is extremum if $\left(\dfrac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0,$ since for $\alpha = 0$ the neighboring curve coincides with the curve which gives extremum values of I.

Thus
$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0, \Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right) dx = 0.$$

Integrating the second integration by parts, we get

$$\int_{x}^{x_2} \frac{\partial f}{\partial y}\eta(x)\, dx + \left(\frac{\partial f}{\partial y'}\eta(x)\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx = 0 \qquad \ldots (5)$$

As y is prescribed at the end points, hence on using equations (3) we obtain

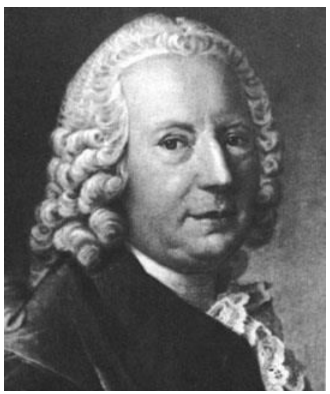
$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta(x)\, dx = 0.$$

Calculus of variation

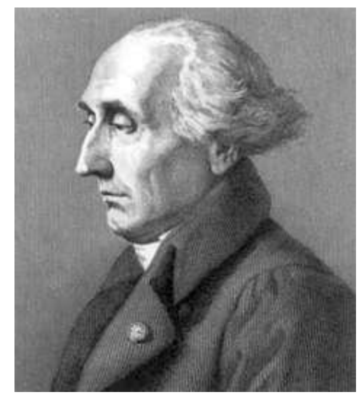
By using the basic lemma of calculus of variation we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad \ldots (6)$$

This is required Euler- Lagrange differential equation to be satisfied by y(x) for which the functional I has extremum value.



Leonhard Euler
(1707 – 1783)



Joseph Louis
Lagrange
(1736 – 1813)

Application of variational principle

1.Shortest distance between two points in a plane

A. Eucledian Plane

Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed points in a Euclidean plane.

Let $y = f(x)$ be the curve between P and Q. Then the element of distance between two neighboring points on the curve $y = f(x)$ joining P and Q is given by

$$ds^2 = dx^2 + dy^2$$

Hence the total distance between the point P and Q along the curve is given by

$$I = \int_P^Q ds$$

$$\Rightarrow \quad I = \int_{x_1}^{x_2} \left(1 + y'^2\right)^{\frac{1}{2}} dx, \quad y' = \frac{dy}{dx} \qquad \ldots (1)$$

Application of variational principle

Here the functional I is extremum if the integrand

$$f = \left(1 + y'^2\right)^{\frac{1}{2}} \qquad \ldots (2)$$

must satisfy the Euler-Lagrange's differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots (3)$$

Now from equation (2) we find that

$$\frac{\partial f}{\partial y} = 0 \quad and \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0.$$

Integrating we get

$$y' = c\sqrt{1+y'^2}\ .$$

Application of variational principle

Squaring we get

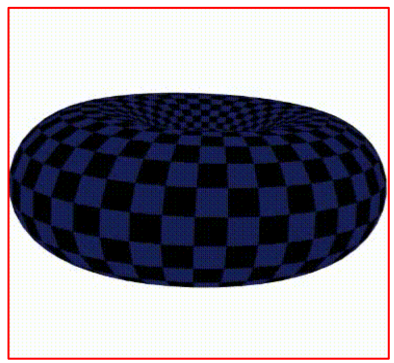
$$y' = c_1, \quad where \quad c_1 = \frac{c}{\sqrt{1+c^2}}.$$

Integrating we get

$$y = c_1 x + c_2. \qquad \ldots (4)$$

Thus the shortest distance between two points in an Eucledian space is a straight line.

Geodesic: A curve representing the shortest distance between any two points on a given surface. The term "geodesic" comes from geodesy, the science of measuring the size and shape of Earth.



If an insect is placed on a surface and continually walks "forward", by definition it will trace out a geodesic.

https://en.wikipedia.org/wiki/Geodesic#/media/File:Insect_on_a_torus_tracing_out_a_non-trivial_geodesic.gif

Application of variational principle

B. Polar Plane

Define a curve in a plane. If A (x, y) and B$(x+dx, y+dy)$ are infinitesimal points on the curve, then an element of distance between A and B is given by

$$ds^2 = dx^2 + dy^2. \qquad \ldots (1)$$

Let $\theta = \theta(r)$ be the polar equation of the curve and $P(r_1, \theta_1)$ *and* $Q(r_2, \theta_2)$ be two polar points on it. Recall the relations

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

Hence equation (1) becomes

$$ds^2 = dr^2 + r^2 d\theta^2. \qquad \ldots (2)$$

Application of variational principle

Thus the total distance between the points P and Q becomes

$$I = \int_{r_1}^{r_2} \left(1 + r^2\theta'^2\right)^{\frac{1}{2}} dr, \quad \theta' = \frac{d\theta}{dr}. \qquad \ldots (3)$$

The functional I is shortest if the integrand

$$f = \left(1 + r^2\theta'^2\right)^{\frac{1}{2}} \qquad \ldots (4)$$

must satisfy the Euler-Lagrange's differential equation

$$\frac{\partial f}{\partial \theta} - \frac{d}{dr}\left(\frac{\partial f}{\partial \theta'}\right) = 0, \qquad (5)$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}}\right) = 0,$$

$$\Rightarrow \quad r^2\theta' = h\sqrt{1 + r^2\theta'^2}.$$

Application of variational principle

Squaring and solving for θ' we get

$$\frac{d\theta}{dr} = \pm \frac{h}{r\left(r^2 - h^2\right)^{\frac{1}{2}}}.$$

On integrating we get

$$\theta = \pm \cos^{-1}\left(\frac{h}{r}\right) + \theta_0,$$

where θ_0 is a constant of integration. We write this as

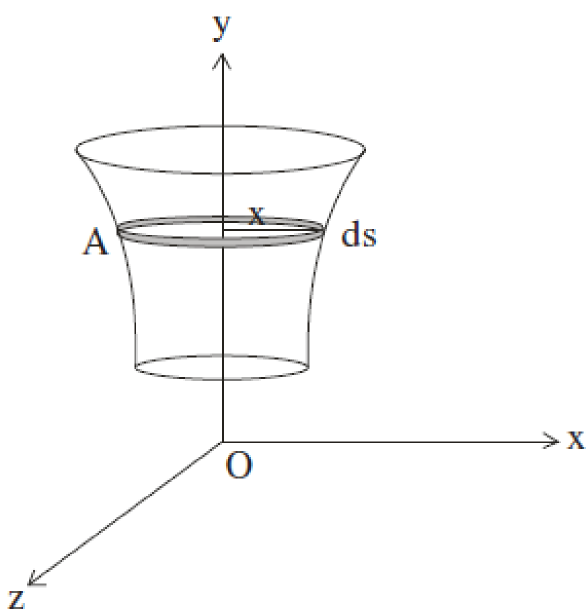
$$h = r\cos(\theta - \theta_0)\,. \qquad \ldots (6)$$

This is the polar form of the equation of straight line. Hence the shortest distance between two polar points is a straight line.

Application of variational principle

2. Minimum surface of revolution

Consider a curve between two points (x_1, y_1) and (x_2, y_2) in the xy plane whose equation is $y = y(x)$. We form a surface by revolving the curve about y-axis. Our claim is to find the nature of the curve for which the surface area is minimum. Consider a small strip at a point A formed by revolving the arc length ds about y –axis. If the distance of the point A on the curve from y-axis is x, then the surface area of the strip is equal to $2\pi x\, ds$.



But we know the element of arc ds is given by

$$ds = \sqrt{1 + y'^2}\, dx .$$

Application of variational principle

Thus the surface area of the strip ds is equal to

$$2\pi x\sqrt{1+y'^2}\,dx\,.$$

Hence the total area of the surface of revolution of the curve $y = y(x)$ about y- axis is given by

$$I = \int_{x_1}^{x_2} 2\pi x\sqrt{1+y'^2}\,dx\,. \qquad \ldots (1)$$

This surface area will be minimum if the integrand

$$f = 2\pi x\sqrt{1+y'^2} \qquad \ldots (2)$$

must satisfy Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \qquad \ldots (3)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{2\pi x\, y'}{\sqrt{1+y'^2}}\right) = 0, \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{x\, y'}{\sqrt{1+y'^2}}\right) = 0\,.$$

Application of variational principle

Integrating we get

$$xy' = a\sqrt{1+y'^2}\,.$$

Solving for y' we get

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}\,.$$

Integrating we get

$$y = a\cosh^{-1}\left(\frac{x}{a}\right) + b\,.$$

Or $$x = a\cosh\left(\frac{y-b}{a}\right). \qquad \ldots (4)$$

This shows that the curve is the catenary.

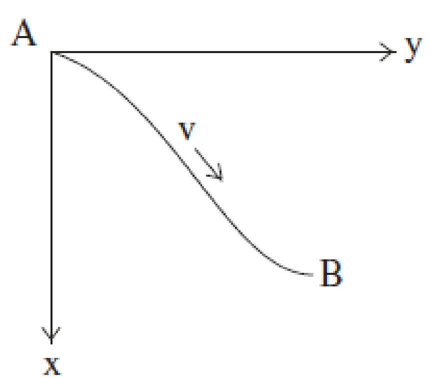
Application of variational principle

3. Brachistochrone problem

To find a curve joining two points along which a particle falling from rest under the influence of gravity travels from higher to the lower point in the minimum time.

Let A and B be two points on the curve not lie on the vertical line.

$v = \dfrac{ds}{dt}$ be the speed of the particle along the curve. Then the time required to fall an arc length ds is given by



$$dt = \frac{ds}{v}$$

$$\Rightarrow \quad dt = \frac{\sqrt{1+y'^2}}{v}dx.$$

Therefore the total time required for the particle to go from A to B is given by

$$t_{AB} = \int_A^B \frac{\sqrt{1+y'^2}}{v}dx \qquad \ldots (1)$$

Application of variational principle

Since the particle falls freely under gravity, therefore its potential energy goes on decreasing and is given by

$$V = -mgx,$$

and the kinetic energy is given by

$$T = \frac{1}{2}mv^2.$$

Now from the principle of conservation of energy we have

$$T + V = \text{constant}.$$

Initially at point A, we have $x = 0$ and $v = 0$. Hence the constant is zero.

$$\Rightarrow \quad \frac{1}{2}mv^2 = mgx,$$

$$\Rightarrow \quad v = \sqrt{2gx}. \qquad \ldots (2)$$

Application of variational principle

Hence equation (1) becomes

$$t_{AB} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gx}} dx \,. \qquad \ldots (3)$$

Thus t_{AB} is minimum if the integrand

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}, \qquad \ldots (4)$$

must satisfy Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots (5)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{2gx(1+y'^2)}}\right) = 0 \quad \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{x(1+y'^2)}}\right) = 0$$

Integrating we get

$$y' = c\sqrt{x(1+y'^2)} \,.$$

Application of variational principle

Solving it for y' we get

$$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{a-x}}$$

Integrating we get

$$y = \int \frac{\sqrt{x}}{\sqrt{a-x}} dx + b \qquad \ldots (6)$$

Put

$$x = a\sin^2(\theta/2)$$
$$\Rightarrow \quad dx = 2a\sin(\theta/2)\cos(\theta/2)d\theta \qquad \ldots (7)$$

Hence

$$y = a\int \sin^2(\theta/2)d\theta + b.$$

$$\Rightarrow \quad y = \frac{a}{2}(\theta - \sin\theta) + b,$$

Application of variational principle

If $$y = 0, \theta = 0 \Rightarrow b = 0,$$

hence

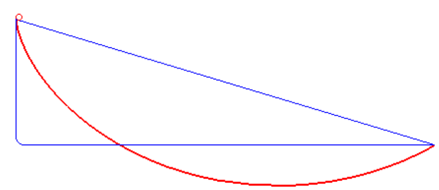
$$y = \frac{a}{2}(\theta - \sin\theta). \qquad \ldots (8)$$

Thus from equations (7) and (8) we have

$$x = b(1 - \cos\theta),$$

$$y = b(\theta - \sin\theta), \quad for \quad b = \frac{a}{2}$$

This is a cycloid. Thus the curve is a cycloid for which the time of decent is minimum.



https://simple.wikipedia.org/wiki/Brachistochrone_curve#/media/File:Brachistochrone.gif

Brachistochrone:
the path of
shortest (brachistos) time
(chronos)

The curve of fastest descent is not a straight or polygonal line (blue) but a cycloid (red).

Suggested readings:

1. *Classical Mechanics*, H. Goldstein, C.P. Poole, J.L. Safko, 3rd Edn. 2002, Pearson Education
2. *Mechanics*, L. D. Landau and E. M. Lifshitz, 1976, Pergamon