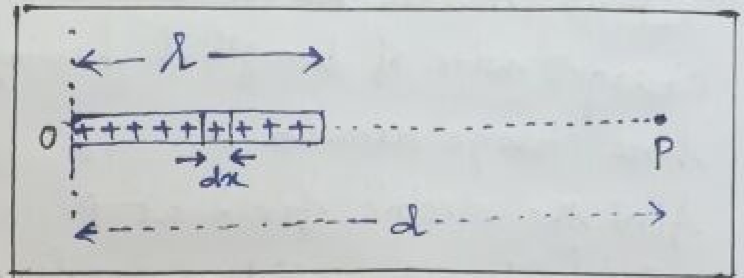
Electricity and Magnetism

* Electric field due to a finite line of charge:

A) On a axial point :



Let us consider an uniformly charged wire of length l having line charge density λ.

Now to find the electric field at any point P on the axis of the wire which is at a distance d, let us consider a small elementary length dx at a distance x.

Now the electric field at P due to dx is

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{(d-x)^2}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dx}{(d-x)^2} \quad , \text{ along } \vec{OP}$$

So, net electric field due the whole wire is

$$E = \frac{\lambda}{4\pi\epsilon_0} \int_0^l \frac{dx}{(d-x)^2}$$

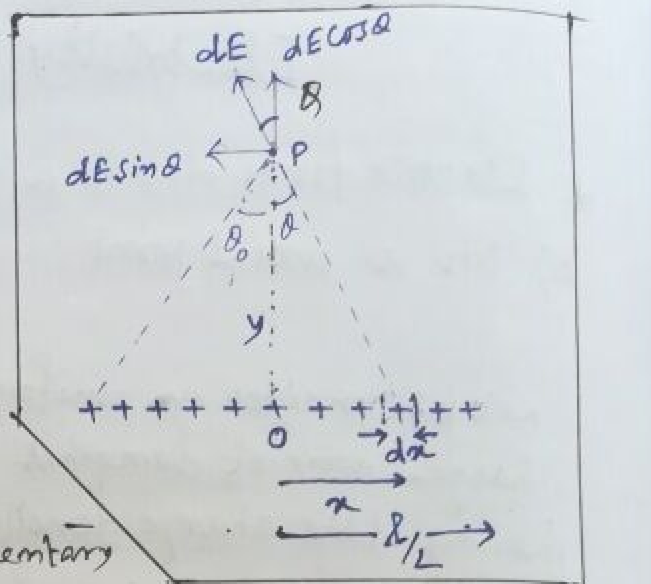
$$= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{d-x} \right]_0^l$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{d-l} - \frac{1}{d} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda l}{d(d-l)}$$

The direction of the electric field is along $\vec{OP}$

B) On a equatorial point :.



- Let us consider an uniformly charged wire of length l having line charge density λ. To find the electric field at P which is at a distance y from O let us consider a small elementary length dx at a distance x from O.

Now the electric field due to this elementary length is

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dx}{x^2+y^2}$$

The two component of dE is $dE\cos\theta$ and $dE\sin\theta$.

Considering the whole wire $dE\sin\theta$ components are cancelled.

so, net electric field at P is

$$E = \int dE\cos\theta = \int_{-l/2}^{+l/2} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dx}{x^2+y^2} \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{2\lambda y}{4\pi\epsilon_0} \int_0^{l/2} \frac{dx}{(x^2+y^2)^{3/2}}$$

$$= \frac{2\lambda y}{4\pi\epsilon_0} \int_0^{\theta_0} \frac{y\sec^2\theta\, d\theta}{y^3 \times \sec^3\theta}$$

Put $x = y\tan\theta$

$dx = y\sec^2\theta\, d\theta$.

x	0	$l/2$
θ	0	θ_0

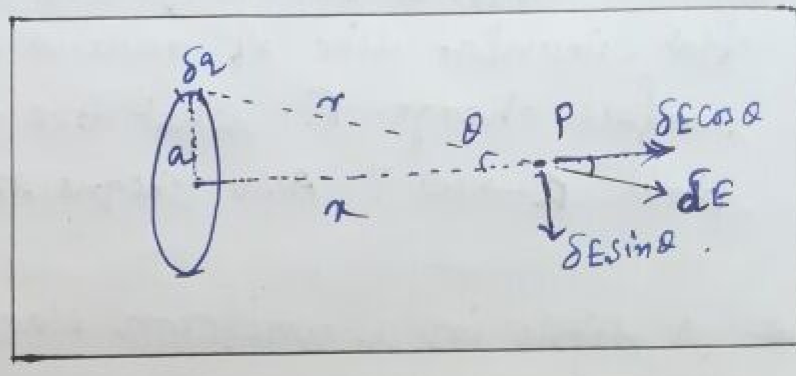
$$= \frac{2\lambda}{4\pi\epsilon_0 y} \int_0^{\theta_0} \cos\theta\, d\theta$$

$$= \frac{2\lambda}{4\pi\epsilon_0} \sin\theta_0$$

$$= \frac{2\lambda}{4\pi\epsilon_0 y} \times \frac{l}{2\sqrt{y^2+(l/2)^2}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda l}{y\sqrt{y^2+(l/2)^2}}$$

The direction of field is along $\vec{PB}$.

* Electric field on the axis of a uniformly charged ring.



Let us consider a uniformly charged ring of radius a whose total charge is q. Let us find the electric field at an axial point P at a distance x from the centre of the ring.

Now due to small charge δq the electric field at P is

$$\delta E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\delta q}{r^2} = \frac{1}{4\pi\epsilon_0}\cdot\frac{\delta q}{(a^2+x^2)}$$

The two components of δE is $\delta E\cos\theta$ and $\delta E\sin\theta$.

Due to the whole ring $\delta E\sin\theta$ components are cancelled

So, net electric field at P is,

$$E = \sum \delta E\cos\theta.$$

$$= \sum \frac{1}{4\pi\epsilon_0}\cdot\frac{\delta q}{(a^2+x^2)}\cdot\frac{x}{\sqrt{a^2+x^2}}$$

$$= \frac{x}{4\pi\epsilon_0(a^2+x^2)^{3/2}}\sum\delta q$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{qx}{(a^2+x^2)^{3/2}}$$

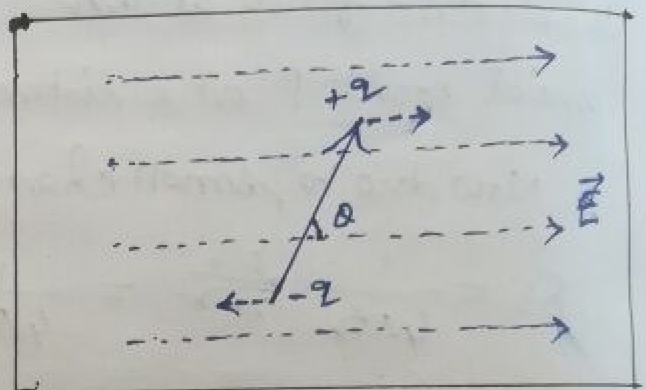
The direction of the field is along $\vec{OP}$.

Special cases :- i) At the centre of the ring i.e at $x=0$, $E=0$

ii) for $x \gg a$, $E \simeq \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{x^2}$

Problem: Find the electric field at a distance x on the axis of a flat circular disc of radius a which carries a uniform surface charge σ. What does your formula give in the limit $a \to \infty$? Also cheque the case $x >> a$.

* A dipole in a uniform electric field :



Let two charges +q and -q separated by a distance 2d constitute an electric dipole of moment $P = q \times 2d$. The dipole is placed in a uniform electric field $\vec{E}$ at an angle θ with the field direction. Due to the two equal and opposite forces acting on two poles a moment of couple i.e. torque is developed. So,

$$\tau = qE \times 2d\sin\theta$$

$$= PE\sin\theta \quad [\because P = q \times 2d]$$

$$\vec{\tau} = \vec{P} \times \vec{E}$$

This amount of torque tends to align the dipole along the field direction.

Now to rotate the dipole by an angle $d\theta$ when it is at an angle θ then a small amount of work is to be done, which is

$$dw = \tau\, d\theta = PE\sin\theta\, d\theta.$$

Thus the work done to rotate the dipole from $0°$ to $\theta°$ is

$$W = \int_0^\theta PE \sin\theta \, d\theta$$

$$= PE(1-\cos\theta)$$

Clearly this amount of work done is stored up in the form of potential energy in the final position.

~~Special cases~~.

The work done to rotate the dipole from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} PE\sin\theta \, d\theta = PE(\cos\theta_1 - \cos\theta_2)$$

Now potential of the dipole at any angle θ is given by

$$W = \int_{90°}^{\theta} PE \sin\theta \, d\theta$$

$$= -PE\cos\theta$$

$$= -\vec{P}.\vec{E}$$

Hence, $U_{\theta=0} = -PE$

$U_{\theta=\pi} = +PE$.

Thus ~~θ=0~~ at $\theta = 0$, the dipole has lowest potential, hence it gives stable equilibrium. $\theta = \pi$ gives unstable equilibrium.

* A dipole in a non-uniform electric field :

Let us consider a small dipole placed in a non-uniform electric field. Due to variation of the field, forces acting on two charges are not equal. So the dipole experiences a net translatory force in addition to the torque.

Let E_x, E_y and E_z be the component of E at the position of $-q$ charge (x, y, z) and E_x', E_y', E_z' be those of E at the position of $+q$ charge $(x+dx,\ y+dy,\ z+dz)$

Then,

$$E_x' = E_x + dE_x$$
$$= E_x + \frac{\partial E_x}{\partial x}.dx + \frac{\partial E_x}{\partial y}dy + \frac{\partial E_x}{\partial z}dz$$

So, in the x direction the net force on the dipole is

$$F_x = qE_x' - qE_x$$
$$= q\left(\frac{\partial E_x}{\partial x}dx + \frac{\partial E_x}{\partial y}dy + \frac{\partial E_x}{\partial z}dz\right)$$

Now, $P_x = q\,dx$, $P_y = q\,dy$, $P_z = q\,dz$.

$$\therefore F_x = P_x\frac{\partial E_x}{\partial x} + P_y\frac{\partial E_x}{\partial y} + P_z\frac{\partial E_x}{\partial z}$$

$$\text{or, } F_x = \left(P_x\hat{i} + P_y\hat{j} + P_z\hat{k}\right).\left(\hat{i}\frac{\partial E_x}{\partial x} + \hat{j}\frac{\partial E_x}{\partial y} + \hat{k}\frac{\partial E_x}{\partial z}\right)$$
$$= \vec{P}.\vec{\nabla}E_x$$

Similarly, the y and z component of net force acting on the dipole is

$$F_y = \vec{P}.\vec{\nabla}E_y$$

$$\text{and, } F_z = \vec{P}.\vec{\nabla}E_z$$

So, the total translatory force experienced by a dipole in a non-uniform electric field is

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$= \vec{p}.\vec{\nabla} E_x \hat{i} + \vec{p}.\vec{\nabla} E_y \hat{j} + \vec{p}.\vec{\nabla} E_z \hat{k}$$

$$= (\vec{p}.\vec{\nabla})\vec{E}$$

where $\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$