3.6: Non-homogeneous equations; method of undetermined coefficients

Want to solve the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{N}$$

Steps:

1. First solve the homogeneous equation

   $$y'' + p(t)y' + q(t)y = 0, \tag{H}$$

   i.e., find y_1, y_2, linearly independent of each other, and form the general solution

   $$y_H = c_1 y_1 + c_2 y_2.$$

2. Find a particular/specific solution Y for (N), by MUC (method of undetermined coefficients);

3. The general solution for (N) is then

   $$y = y_H + Y = c_1 y_1 + c_2 y_2 + Y.$$

   Find c_1, c_2 by initial conditions, if given.

Key step: step 2.

Why $y = y_H + Y$?
A quick proof: If y_H solves (H), then

$$y_H'' + p(t)y_H' + q(t)y_H = 0, \tag{A}$$

and since Y solves (N), we have

$$Y'' + p(t)Y' + q(t)Y = g(t), \tag{B}$$

Adding up (A) and (B), and write $y = y_H + Y$, we get $y'' + p(t)y' + q(t)y = g(t)$.
Main focus: constant coefficient case, i.e.,

$$ay'' + by' + cy = g(t).$$

Example 1. Find the general solution for $\quad y'' - 3y' + 4y = 3e^{2t}$.

Answer. Step 1: Find y_H.

$$r^2 - 3r - 4 = (r+1)(r-4) = 0, \quad \Rightarrow \quad r_1 = -1,\ r_2 = 4,$$

so

$$y_H = c_1 e^{-t} + c_2 e^{4t}.$$

Step 2: Find Y. We guess/seek solution of the same form as the source term $Y = Ae^{2t}$, and will determine the coefficient A.

$$Y' = 2Ae^{2t}, \quad Y'' = 4Ae^{2t}.$$

Plug these into the equation:

$$4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4Ae^{2t} = 3e^{2t}, \quad \Rightarrow \quad -6A = 3, \quad \Rightarrow \quad A = -\frac{1}{2}.$$

So $Y = -\frac{1}{2}e^{2t}$.

Step 3. The general solution to the non-homogeneous solution is

$$y(t) = y_H + Y = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t}.$$

Observation: The particular solution Y take the same *form* as the source term $g(t)$.

But this is not always true.

Example 2. Find general solution for $\quad y'' - 3y' + 4y = 2e^{-t}$.

Answer. The homogeneous solution is the same as Example 1: $y_H = c_1 e^{-t} + c_2 e^{4t}$. For the particular solution Y, let's first try the same form as g, i.e., $Y = Ae^{-t}$. So $Y' = -Ae^{-t}, Y'' = Ae^{-t}$. Plug them back in to the equation, we get

$$\text{LHS} = Ae^{-t} - 3(-Ae^{-t}) - 4Ae^{-t} = 0 \neq 2e^{-et} = \text{RHS}.$$

So it doesn't work. Why?

We see $r_1 = -1$ and $y_1 = e^{-t}$, which means our guess $Y = Ae^{-t}$ is a solution to the homogeneous equation. It will never work.

Second try: $Y = Ate^{-t}$. So

$$Y' = Ae^{-t} - Ate^{-t}, \quad Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}.$$

Plug them in the equation

$$(-2Ae^{-t} + Ate^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} = -5Ae^{-t} = 2e^{-t},$$

we get

$$-5A = 2, \quad \Rightarrow \quad A = -\frac{2}{5},$$

so we have $Y = -\frac{2}{5}te^{-t}$.

Summary 1. If $g(t) = ae^{\alpha t}$, then the form of the particular solution Y depends on r_1, r_2 (the roots of the characteristic equation).

case	form of the particular solution Y
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$Y = Ae^{\alpha t}$
$r_1 = \alpha$ or $r_2 = \alpha$, but $r_1 \neq r_2$	$Y = Ate^{\alpha t}$
$r_1 = r_2 = \alpha$	$Y = At^2e^{\alpha t}$

Example 3. Find the general solution for

$$y'' - 3y' - 4y = 3t^2 + 2.$$

Answer. The y_H is the same $y_H = c_1e^{-t} + c_2e^{4t}$.
Note that $g(t)$ is a polynomial of degree 2. We will try to guess/seek a particular solution of the same form:

$$Y = At^2 + Bt + C, \qquad Y' = 2At + B, \qquad Y'' = 2A$$

Plug back into the equation

$$2A - 3(2At + b) - 4(At^2 + Bt + C) = -4At^2 - (6A + 4B)t + (2A - 3B - 4C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -4A = 3 \qquad &\to \quad A = -\frac{3}{4} \\ -(6A+4B) = 0, \quad &\to \quad B = \frac{9}{8} \\ 2A - 3B - 4C = 2, \quad &\to \quad C = \frac{1}{4}(2A - 3B - 2) = -\frac{55}{32} \end{aligned}$$

So we get

$$Y(t) = -\frac{3}{4}t^2 + \frac{9}{8}t - \frac{55}{32}.$$

But sometimes this guess won't work.

Example 4. Find the particular solution for $y'' - 3y' = 3t^2 + 2$.

Answer. We see that the form we used in the previous example $Y = At^2 + Bt + C$ won't work because $Y'' - 3Y'$ will not have the term t^2.

New try: multiply by a t. So we guess $Y = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$. Then

$$Y' = 3At^2 + 2Bt + C, \quad Y'' = 6At + 2B.$$

Plug them into the equation

$$(6At + 2B) - 3(3At^2 + 2Bt + C) = -9At^2 + (6A - 6B)t + (2B - 3C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -9A = 3 \qquad &\to \quad A = -\frac{1}{3} \\ (6A - 6B) = 0, \quad &\to \quad B = A = -\frac{1}{3} \\ 2B - 3C = 2, \quad &\to \quad C = \frac{1}{3}(2B - 2) = -\frac{8}{9} \end{aligned}$$

So $Y = t(-\frac{1}{3}t^2 - \frac{1}{3}t - \frac{8}{9})$.

Summary 2. If $g(t)$ is a polynomial of degree n, i.e.,

$$g(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$$

the particular solution for

$$ay'' + by' + cy = g(t)$$

(where $a \neq 0$) depends on b, c:

case	form of the particular solution Y
$c \neq 0$	$Y = P_n(t) = A_n t^n + \cdots + A_1 t + A_0$
$c = 0$ but $b \neq 0$	$Y = tP_n(t) = t(A_n t^n + \cdots + A_1 t + A_0)$
$c = 0$ and $b = 0$	$Y = t^2 P_n(t) = t^2(A_n t^n + \cdots + A_1 t + A_0)$

Example 5. Find a particular solution for

$$y'' - 3y' - 4y = \sin t.$$

Answer. Since $g(t) = \sin t$, we will try the same form. Note that $(\sin t)' = \cos t$, so we must have the $\cos t$ term as well. So the form of the particular solution is

$$Y = A \sin t + B \cos t.$$

Then

$$Y' = A \cos t - B \sin t, \qquad Y'' = -A \sin t - B \cos t.$$

Plug back into the equation, we get

$$\begin{aligned} & (-A \sin t - B \cos t) - 3(A \cos t - B \sin t) - 4(A \sin t + b \cos t) \\ = \quad & (-5A + 3B) \sin t + (-3A - 5B) \cos t = \sin t. \end{aligned}$$

So we must have

$$-5A + 3B = 1, \quad -3A - 5B = 0, \qquad \rightarrow \quad A = \frac{5}{34}, \quad B = \frac{3}{34}.$$

So we get

$$Y(t) = -\frac{5}{34} \sin t + \frac{3}{34} \cos t.$$

But this guess won't work if the form is a solution to the homogeneous equation.

Example 6. Find a general solution for $y'' + y = \sin t$.

Answer. Let's first find y_H. We have $r^2 + 1 = 0$, so $r_{1,2} = \pm i$, and $y_H = c_1 \cos t + c_2 \sin t$.

For the particular solution Y: We see that the form $Y = A \sin t + B \cos t$ won't work because it solves the homogeneous equation.

Our new guess: multiply it by t, so

$$Y(t) = t(A \sin t + B \cos t).$$

Then

$$Y' = (A \sin t + B \cos t) + t(A \cos t + B \sin t),$$

$$Y'' = (-2B - At) \sin t + (2A - Bt) \cos t.$$

Plug into the equation

$$Y'' + Y = -2B \sin t + 2A \cos t = \sin t, \quad \Rightarrow \quad A = 0, \quad B = -\frac{1}{2}$$

So

$$Y(y) = -\frac{1}{2} t \cos t.$$

The general solution is

$$y(t) = y_H + Y = c_1 \cos t + c_2 \sin t - \frac{1}{2} t \cos t.$$

Summary 3. If $g(t) = a \sin \alpha t + b \cos \alpha t$, the form of the particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
$r_{1,2} \neq \pm \alpha i$	$Y = A \sin \alpha t + B \cos \alpha t$
$r_{1,2} = \pm \alpha i$	$Y = t(A \sin \alpha t + B \cos \alpha t)$

Next we study a couple of more complicated forms of g.

Example 7. Find a particular solution for

$$y'' - 3y' - 4y = te^t.$$

Answer. We see that $g = P_1(t)e^{at}$, where P_1 is a polynomial of degree 1. Also we see $r_1 = -1, r_2 = 4$, so $r_1 \neq a$ and $r_2 \neq a$. For a particular solution we will try the same form as g, i.e., $Y = (At + B)e^t$. So

$$Y' = Ae^t + (At + b)e^t = (A + b)e^t + Ate^t,$$

$$Y'' = \cdots = (2A + B)e^t + Ate^t.$$

Plug them into the equation,

$$[(2A+B)e^t + Ate^t] - 3[(A+b)e^t + Ate^t] - 4(At+B)e^t = (-6At - A - 6B)e^t = te^t.$$

We must have $-6At - A - 6B = t$, i.e.,

$$-6A = 1, \quad -A - 6B = 0, \quad \Rightarrow \quad A = -\frac{1}{6}, B = \frac{1}{36}, \quad \Rightarrow \quad Y = (-\frac{1}{6}t + \frac{1}{36})e^t.$$

However, if the form of g is a solution to the homogeneous equation, it won't work for a particular solution. We must multiply it by t in that case.

Example 8. Find a particular solution of

$$y'' - 3y' - 4y = te^{-t}.$$

Answer. Since $a = -1 = r_1$, so the form we used in Example 7 won't work here. Try

$$Y = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}.$$

Then

$$Y' = \cdots = [-At^2 + (2A - B)t + B]e^{-t},$$

$$Y'' = \cdots = [At^2 + (B - 4A)t + 2A - 2B]e^{-t}.$$

Plug into the equation

$$[At^2 + (B - 4A)t + 2A - 2B]e^{-t} - 3[-At^2 + (2A - B)t + B]e^{-t} - 4(At^2 + Bt)e^{-t}$$
$$= \ [-10At + 2A - 5B]e^{-t} \ = \ te^t.$$

So we must have $-10At + 2A - 5B = t$, which means

$$-10A = 1, \quad 2A - 5B = 0, \qquad \Rightarrow \qquad A = -\frac{1}{10}, \quad B = -\frac{1}{25}.$$

Then

$$Y = \left(-\frac{1}{10}t^2 - \frac{1}{25}t\right) e^{-t}.$$

Summary 4. If $g(t) = P_n(t)e^{at}$ where $P_n(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$ is a polynomial of degree n, then the form of a particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
$r_1 \neq a$ and $r_2 \neq a$	$Y = \tilde{P}_n(t)e^{at} = (A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = a$ or $r_2 = a$ but $r_1 \neq r_2$	$Y = t\tilde{P}_n(t)e^{at} = t(A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = r_2 = a$	$Y = t^2\tilde{P}_n(t)e^{at} = t^2(A_n t^n + \cdots + A_1 t + A_0)e^{at}$

Other cases of g are treated in a similar way: Check if the form of g is a solution to the homogeneous equation. If not, then use it as the form of a particular solution. If yes, then multiply it by t or t^2.

We summarize a few cases below.

Summary 5. If $g(t) = e^{\alpha t}(a \cos \beta t + b \sin \beta t)$, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}(A \cos \beta t + B \sin \beta t)$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}(A \cos \beta t + B \sin \beta t)$

Summary 6. If $g(t) = P_n(t)e^{\alpha t}(a \cos \beta t + b \sin \beta t)$ where $P_n(t)$ is a polynomial of degree n, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$

If the source $g(t)$ has several terms, we treat each separately and add up later. Let $g(t) = g_1(t) + g_2(t) + \cdots g_n(t)$, then, find a particular solution Y_i for each $g_i(t)$ term as if it were the only term in g, then $Y = Y_1 + Y_2 + \cdots Y_n$. This claim follows from the principle of superposition.

In the examples below, we want to write the form of a particular solution.

Example 9. $\quad y'' - 3y' - 4y = \sin 4t + 2e^{4t} + e^{5t} - t.$

Answer. Since $r_1 = -1, r_2 = 2$, we treat each term in g separately and the add up:

$$Y(t) = A\sin 4t + B\cos 4tCte^{4t} + De^{5t} + (Et + F).$$

Example 10. $\quad y'' + 16y = \sin 4t + \cos t - 4\cos 4t + 4.$

Answer. The char equation is $r^2 + 16 = 0$, with roots $r_{1,2} = \pm 4i$, and

$$y_H = c_1 \sin 4t + c_2 \cos 4t.$$

We also note that the terms $\sin 4t$ and $-4\cos 4t$ are of the same type, and we must multiply it by t. So

$$Y = t(A\sin 4t + B\cos 4t) + (C\cos t + D\sin t) + E.$$

Example 11. $\quad y'' - 2y' + 2y = e^t \cos t + 8e^t \sin 2t + te^{-t} + 4e^{-t} + t^2 - 3.$

Answer. The char equation is $r^2 - 2r + 2 = 0$ with roots $r_{1,2} = 1 \pm i$. Then, for the term $e^t \cos t$ we must multiply by t.

$$Y = te^t(A_1\cos t + A_2\sin t) + e^t(B_1\cos 2t + B_2\sin 2t) + (C_1 t + C_0)e^{-t} + De^{-t} + (F_2 t^2 + F_1 t + F_0).$$