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"Network theorem"

By

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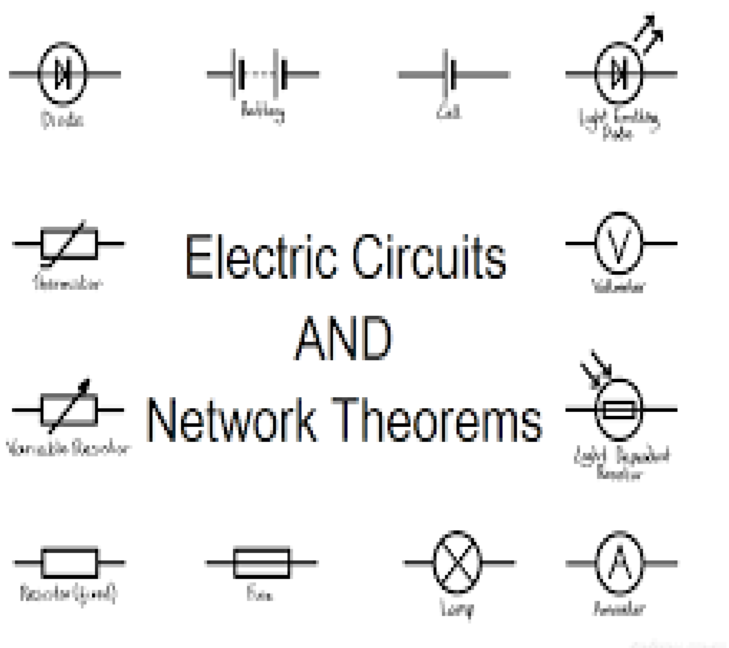


This is a little approach to ensure the access of study materials for the Physics (Hons.) students of 2nd Semester, Dept. of Physics,RaniganjGirls'College (Aff. To Kazi Nazrul university) during the emergency "COVID-19 pandemic lockdown" period .

The study material is being prepared as per the Syllabus of Kazi Nazrul university.

Course details* :** This chapter is under the "Core Course-4: Electricity and magnetism".***

Syllabus

Nerwork theorems: Ideal Constant-voltage and constant-current Sources, Network theorems: Thevenin theorem, Norton theorem, Superposition theorem, Reciprocity theorem, Maximum power transfer theorem. Applications to dc circuits.

***Reference* :**

1. A TEXTBOOK OF ELECTRICAL TECHNOLOGY: VOL- I: B.L. THEREJA, A.K. THERAJA
2. ELECTRICITY AND MAGNETISM: D.CHATTOPADHYAY AND P.C. RAKSHIT
3. ELECTRICITY AND MAGNETISM: MAHAJAN AND RANGWALA

Few important terminologies:

1. **Circuit:** A circuit is a closed conducting path through which an electric current either flows or is intended flow.
2. **Parameters:** The various elements of an electric circuit are called its parameters like resistance, inductance and capacitance.
3. **Linear Circuit:** A linear circuit is one whose parameters are constant *i.e.* they do not change with voltage or current.
4. **Non-linear Circuit:** It is that circuit whose parameters change with voltage or current.
5. **Bilateral Circuit:** A bilateral circuit is one whose properties or characteristics are the same in either direction. The usual transmission line is bilateral, because it can be made to perform its function equally well in either direction.
6. **Unilateral Circuit:** It is that circuit whose properties or characteristics change with the direction of its operation. A diode rectifier is a unilateral circuit, because it cannot perform rectification in both directions.
7. **Electric Network:** A combination of various electric elements, connected in any manner whatsoever, is called an electric network.
8. **Passive Network:** It is one which contains no source of e.m.f. in it.
9. **Active Network:** It is one which contains one or more than one source of e.m.f.
10. **Node:** It is a junction in a circuit where two or more circuit elements are connected together.
11. **Branch:** It is that part of a network which lies between two junctions.
12. **Loop:** It is a close path in a circuit in which no element or node is encountered more than once.
13. **Mesh:** It is a loop that contains no other loop within it.

Now, we have to introduce some basic laws to understand the network theorems which are going to be used for the analytical solution of problems related to complex electrical networks.

Ohm's law: For a conductor, such as a metal, a linear relationship exists between the potential difference of the two ends of the conductor and the current flowing between them provided the temperature and the other physical conditions remain unchanged. This is the so called **ohm's law**.

For a straight conducting wire of uniform cross-sectional area, if V be the potential difference between any two points of the wire, Ohm's law takes the form:

$$V=IR \qquad \text{---------------------} \qquad \textbf{(1)}$$

Where, I is the current flowing through the two points of the wire and R is the proportionality constant, called the resistance of the wire. The resistance has the unit of volt/ampere or ohm (Ω) in SI. If l is the length and A is the cross-sectional area of the wire, the resistance R is given by, $R=\rho l/A$ ---------------------- **(2)**

Where, ρ is a constant depending on the material of the wire, called the resistivity of the material and has the unit of ohm-meter.

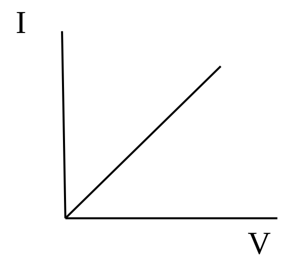
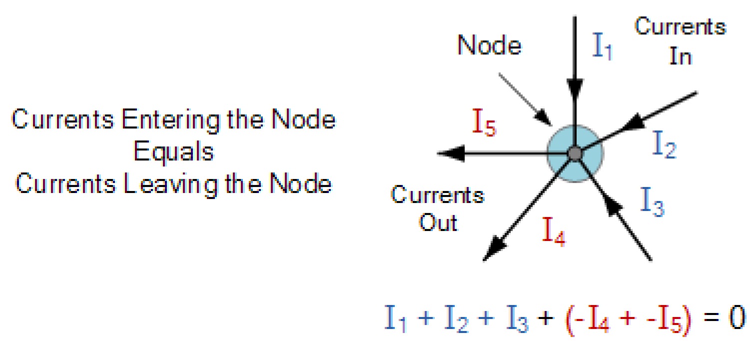


Fig: V-I characteristics for ohmic devices.

Kirchhoff's laws: Kirchhoff's laws offer a convenient expression for the solution of dc circuit problems. These laws can be stated in the following way:

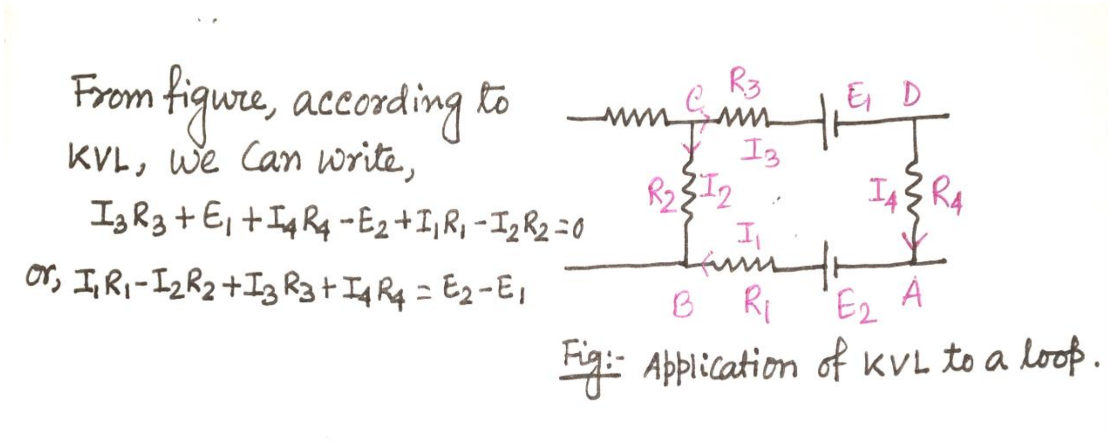
1. **Kirchhoff's current law (KCL):** The algebraic sum of all the currents at any junction is zero.

   If a current flowing towards a junction is taken as positive, a current flowing out of the junction must be taken as negative.



   It is a consequence of conservation of charge.

2. **Kirchhoff's voltage law (KVL):** The sum of the IR drops around any loop is equal to the total emf acting in that loop.



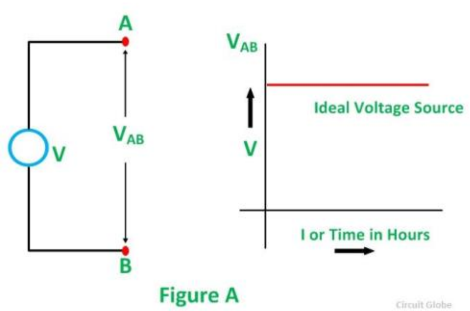
It is a consequence of conservation of energy.

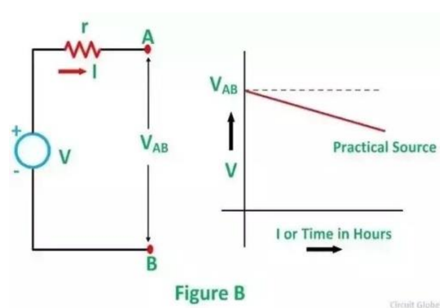
<u>Ideal Constant-voltage source and ideal constant-current source</u>:

An ideal voltage source is a voltage generator whose output voltage is independent of the current delivered by the generator. The symbol for an ideal voltage source and its current-voltage characteristics is shown in figure-A.

An ideal voltage source must have zero internal impedance. A practical voltage source always has some internal impedance.

So, a practical voltage source can be represented by an ideal voltage source in series with a resistance or an impedance, which is shown in figure-B.





An ideal current source is a current generator which supplies a current independent of the voltage across the terminals of the current generator. The symbol of an ideal current source and its' current-voltage characteristics is shown in figure-C.

An ideal current source must have infinite internal impedance.

A practical current source can be represented by an ideal current source in parallel with an internal impedance as shown in figure-D.

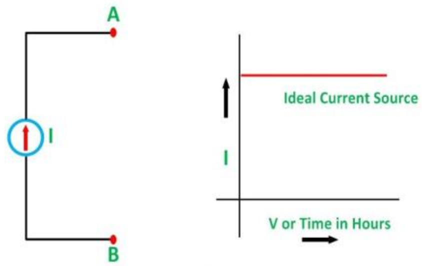


Figure C

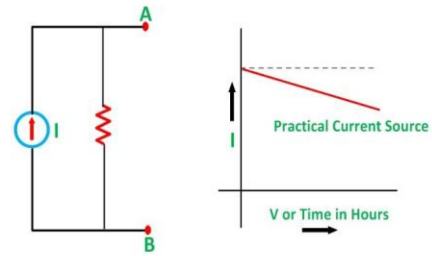


Figure D

Network theorems:-

Thevenin's theorem:-

According to this theorem, in any two terminal linear network containing energy sources and resistances, the current flowing through a load resistance is the same as that supplied by a single generator of open circuit voltage V_T, together with a series resistance R_T.

The voltage V_T is the open circuit voltage between the terminals of the network to which the load resistance is connected and R_T is the resistance of the circuit looking back from these terminals, with all the energy sources are replaced by their internal resistances.

Proof:-

To prove this theorem, let us consider the following two port network and then apply KVL to the well defined loops

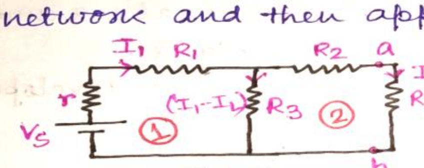


Fig-I

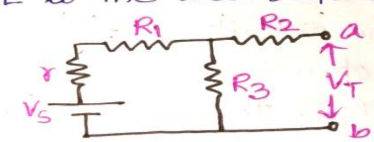


Fig-II

Applying KVL, in loop ①,

$$I_1R_1 + I_1R_3 - V_S + I_1r - I_LR_3 = 0$$

$$\Rightarrow (R_1+R_3+r)I_1 - R_3I_L = V_S \quad \text{——— (1)}$$

and in loop ②,

$$I_LR_L + I_LR_3 + I_LR_2 - I_1R_3 = 0$$

$$\Rightarrow -R_3I_1 + (R_2+R_3+R_L)I_L = 0 \quad \text{——— (2)}$$

∴ Current through the load resistance R_L is,

$$I_L = \frac{\begin{vmatrix} (R_1+R_3+r) & V_S \\ -R_3 & 0 \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r) & -R_3 \\ -R_3 & (R_2+R_3+R_L) \end{vmatrix}}$$

$$\therefore I_L = \frac{V_S R_3}{(R_1+R_3+r)(R_2+R_3+R_L)-R_3^2}$$

$$= \frac{V_S R_3}{(R_2+R_L)(R_1+R_3+r)+R_3(R_1+R_3+r)-R_3^2}$$

$$= \frac{V_S R_3}{(R_2+R_L)(R_1+R_3+r)+R_3(R_1+r)+\cancel{R_3^2}-\cancel{R_3^2}}$$

$$\Rightarrow I_L = \frac{V_S R_3}{(R_2+R_L)(R_1+R_3+r)+R_3(R_1+r)} \quad \text{——— (3)}$$

Step-I

Let us now disconnect the load resistance R_L to determine the open circuit voltage V_T across the terminals a and b.

The circuit of fig-I then reduces to fig-II.

The open circuit voltage is the same as that developed across R_3, the current through R_2 being zero.

$\therefore$ Current through R_3 is,

$$i = \frac{V_S}{(R_1+R_3+r)}$$

$$\therefore V_T = iR_3 = \frac{V_S R_3}{(R_1+R_3+r)} \quad \text{——— (4)}$$

Step-II

To find the Thevenin's resistance R_T, source has been replaced by it's internal resistance 'r'. R_T is then the resistance offered by the circuit of fig-III, across the terminals a and b.

$$R_T = R_2 + \frac{R_3(R_1+r)}{(R_3+R_1+r)}$$

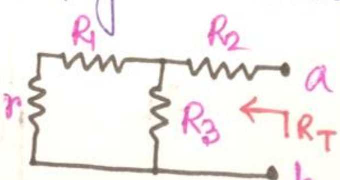


Fig-III

$$\text{or, } R_T = \frac{R_2(R_3+R_1+r)+R_3(R_1+r)}{(R_3+R_1+r)} \quad \text{— (5)}$$

Step-III

Let us now Connect the load resistance R_L across a generator of open circuit voltage V_T and internal resistance R_T as shown in fig-IV.

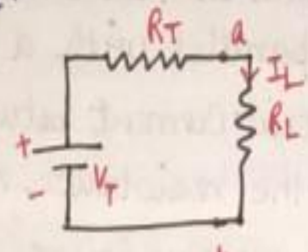


Fig-IV: Thevenin's equivalent circuit.

$\therefore$ Current through the load resistance R_L is,

$$I_L = \frac{V_T}{(R_T+R_L)} = \frac{V_S R_3}{(R_1+R_3+r)} \cdot \frac{1}{(R_T+R_L)} \quad [\text{From}(4)]$$

$$\text{or, } I_L = \frac{V_S R_3}{\frac{R_2(R_3+R_1+r)+R_3(R_1+r)}{(R_3+R_1+r)}+R_L} \cdot \frac{1}{(R_1+R_3+r)} \quad [\text{From}(5)]$$

$$= \frac{V_S R_3 \cancel{(R_3+R_1+r)}}{R_2(R_3+R_1+r)+R_3(R_1+r)+R_L(R_3+R_1+r)} \cdot \frac{1}{\cancel{(R_1+R_3+r)}}$$

$$\Rightarrow I_L = \frac{V_S R_3}{(R_2+R_L)(R_3+R_1+r)+R_3(R_1+r)} \quad \text{---}(6)$$

The equations (3) and (6) are same which establishes the 'Thevenin's theorem'.

Norton's theorem :-

According to this theorem, in a circuit containing linear ohmic resistances and energy sources, the current flowing through a load resistance is the same as that obtained from a single current source I_N in parallel with a resistance R_N. where, I_N is the short-circuit current between the terminals of the network and R_N is the resistance measured between the terminals with all the energy sources are replaced by their internal resistances.

Proof :-

To prove this theorem, let us consider the following two port network as shown in fig-I.

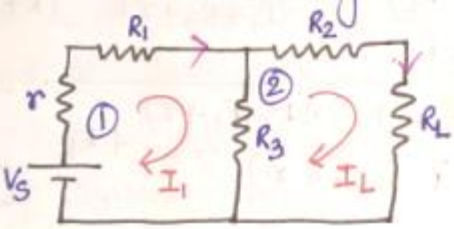


Fig-I

Applying KVL,

From loop (1),

$$I_1R_1 + I_1R_3 - V_S + I_1 r - I_L R_3 = 0$$

$$\Rightarrow (R_1+R_3+r)I_1 - R_3 I_L = V_S \quad \text{—— (1)}$$

and, from loop (2),

$$R_2 I_L + R_L I_L + R_3 I_L - I_1 R_3 = 0$$

$$\Rightarrow -R_3 I_1 + (R_2+R_3+R_L) I_L = 0 \quad \text{—— (2)}$$

$$\therefore I_L = \frac{\begin{vmatrix} (R_1+R_3+r) & V_S \\ -R_3 & 0 \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r) & -R_3 \\ -R_3 & (R_2+R_3+R_L) \end{vmatrix}} = \frac{V_S R_3}{(R_1+R_3+r)(R_2+R_3+R_L) - R_3^2} \quad \text{—— (3)}$$

Step-I

Let us now solve the problem by applying Norton's theorem. In fig-II, the load R_L has been replaced by a connecting wire (conducting), which makes the terminals a and b short circuited.

Applying KVL, we have,

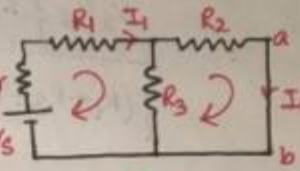


Fig-II

$$R_1 I_1 + R_3 I_1 - V_S + r I_1 - I_N R_3 = 0$$

$$\Rightarrow (R_1+R_3+r) I_1 - R_3 I_N = V_S \quad \text{——— (4)}$$

$$R_2 I_N + R_3 I_N - R_3 I_1 = 0$$

$$\Rightarrow -R_3 I_1 + (R_2+R_3) I_N = 0 \quad \text{——— (5)}$$

Solving these two equation by Cramer's rule,

$$\therefore I_N = \frac{\begin{vmatrix} (R_1+R_3+r) & V_S \\ -R_3 & 0 \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r) & -R_3 \\ -R_3 & (R_2+R_3) \end{vmatrix}} = \frac{V_S R_3}{(R_1+R_3+r)(R_2+R_3) - R_3^2} \quad \text{—— (6)}$$

Step-II

To find Norton's resistance R_N, we consider fig-III, where the source has been replaced by it's internal resistance.

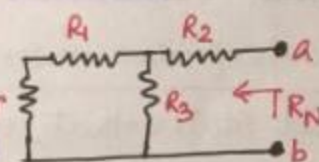


Fig-III

$$\therefore R_N = R_2 + \frac{R_3(R_1+r)}{(R_3+R_1+r)}$$

$$\Rightarrow R_N = \frac{R_2(R_3+R_1+r) + R_3(R_1+r)}{(R_3+R_1+r)} \quad \text{—— (7)}$$

Step-III

Let us now connect the load resistance R_L and the Norton's equivalent circuit is as shown in fig-IV. Voltage across R_N & R_L in parallel,

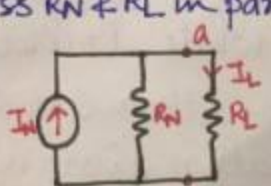


Fig-IV

$$V = I_N \cdot \frac{R_N R_L}{(R_N + R_L)}$$

$\therefore$ Current through R_L,

$$I_L = \frac{V}{R_L} = \frac{I_N R_N}{(R_N + R_L)}$$

$$\therefore I_L = \frac{V_S R_3}{\{(R_1+R_3+r)(R_2+R_3) - R_3^2\}} \times \frac{\{R_2(R_3+R_1+r) + R_3(R_1+r)\}}{(R_3+R_1+r)} \cdot \frac{1}{(R_N+R_L)}$$

$$\therefore I_L = \frac{V_S R_3 \{R_2(R_3+R_1+r) + R_3(R_1+r)\}}{\{(R_1+R_3+r)(R_2+R_3) - R_3^2\} \cdot \cancel{(R_3+R_1+r)}} \cdot \frac{\cancel{(R_3+R_1+r)}}{R_2(R_3+R_1+r) + R_3(R_1+r) + R_L(R_3+R_1+r)}$$

$$= \frac{V_S R_3 \{R_2(R_3+R_1+r) + R_3(R_1+R_3+r) - R_3^2\}}{\{(R_1+R_3+r)(R_2+R_3) - R_3^2\}\{(R_2+R_L)(R_3+R_1+r) + R_3(R_1+r)\}}$$

$$= \frac{V_S R_3 \cancel{\{(R_1+R_3+r)(R_2+R_3) - R_3^2\}}}{\cancel{\{(R_1+R_3+r)(R_2+R_3) - R_3^2\}}\{(R_2+R_L)(R_1+R_3+r) + R_3(R_1+r)\}}$$

$$\text{or, } I_L = \frac{V_S R_3}{(R_1+R_3+r)(R_2+R_L) + R_3(R_1+R_3+r) - R_3^2}$$

$$\Rightarrow I_L = \frac{V_S R_3}{(R_1+R_3+r)(R_2+R_3+R_L) - R_3^2} \quad \text{———} (8)$$

Thus, we see that equations (3) and (8) are identical, which establishes the 'Norton's theorem'.

Superposition theorem :-

It states that, in any linear bilateral network containing several energy sources, the current and voltage in any element is the algebraic sum of the effects produced by each source acting separately.

While considering the effect of a particular source, other sources must be replaced by their internal resistances.

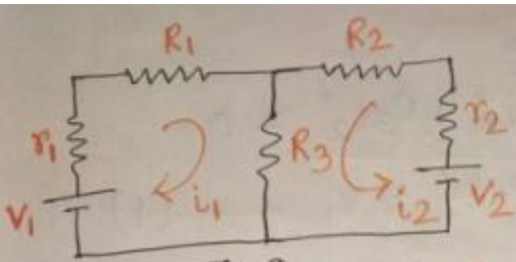


Fig-I

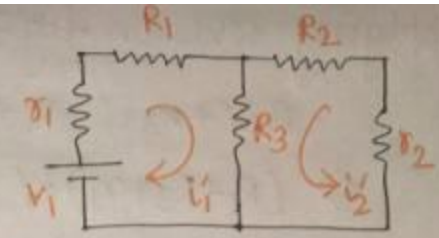


Fig-II

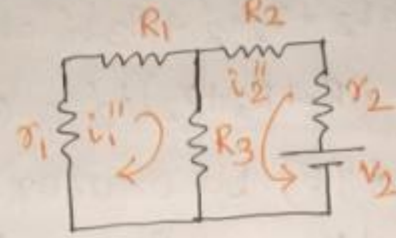


Fig-III

By theorem, $i_1 = i_1' + i_1''$ and $i_2 = i_2' + i_2''$

Proof :-

Applying KVL in fig-I,

$$i_1 R_1 + i_1 R_3 - V_1 + i_1 r_1 + i_2 R_3 = 0$$

$$\Rightarrow (R_1 + R_3 + r_1) i_1 + R_3 i_2 = V_1 \quad \text{——— (1)}$$

$$R_2 i_2 + R_3 i_2 - V_2 + r_2 i_2 + i_1 R_3 = 0$$

$$\Rightarrow R_3 i_1 + (R_2 + R_3 + r_2) i_2 = V_2 \quad \text{——— (2)}$$

∴ Solving equations (1) and (2) by Cramer's rule,

$$i_1 = \frac{\begin{vmatrix} V_1 & R_3 \\ V_2 & (R_2+R_3+r_2) \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r_1) & R_3 \\ R_3 & (R_2+R_3+r_2) \end{vmatrix}}$$

$$\Rightarrow i_1 = \frac{(R_2+R_3+r_2)V_1 - R_3 V_2}{(R_1+R_3+r_1)(R_2+R_3+r_2) - R_3^2} \quad \text{——— (3)}$$

Applying KVL in fig-II,

$$R_1 i_1' + R_3 i_1' - V_1 + r_1 i_1' + R_3 i_2' = 0$$

$$\Rightarrow (R_1+R_3+r_1) i_1' + R_3 i_2' = V_1 \quad \text{——(4)}$$

$$R_2 i_2' + R_3 i_2' + r_2 i_2' + R_3 i_1' = 0$$

$$\Rightarrow R_3 i_1' + (R_2+R_3+r_2) i_2' = 0 \quad \text{——(5)}$$

Solving (4) and (5) by cramer's rule,

$$\therefore i_1' = \frac{\begin{vmatrix} V_1 & R_3 \\ 0 & (R_2+R_3+r_2) \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r_1) & R_3 \\ R_3 & (R_2+R_3+r_2) \end{vmatrix}}$$

$$\text{or, } i_1' = \frac{V_1 (R_2+R_3+r_2)}{(R_1+R_3+r_1)(R_2+R_3+r_2) - R_3^2} \quad \text{——(6)}$$

Also, using KVL in fig-III,

$$R_1 i_1'' + R_3 i_1'' + r_1 i_1'' + R_3 i_2'' = 0$$

$$\Rightarrow (R_1+R_3+r_1) i_1'' + R_3 i_2'' = 0 \quad \text{——(7)}$$

$$R_2 i_2'' + R_3 i_2'' - V_2 + r_2 i_2'' + R_3 i_1'' = 0$$

$$\Rightarrow R_3 i_1'' + (R_2+R_3+r_2) i_2'' = V_2 \quad \text{——(8)}$$

Solving (7) and (8) by cramer's rule,

$$i_1'' = \frac{\begin{vmatrix} 0 & R_3 \\ V_2 & (R_2+R_3+r_2) \end{vmatrix}}{\begin{vmatrix} (R_1+R_3+r_1) & R_3 \\ R_3 & (R_2+R_3+r_2) \end{vmatrix}} = \frac{-V_2 R_3}{(R_1+R_3+r_1)(R_2+R_3+r_2) - R_3^2} \quad (9)$$

From (3),

$$i_1 = \frac{(R_2+R_3+r_2)V_1 - R_3V_2}{(R_1+R_3+r_1)(R_2+R_3+r_2) - R_3^2}$$

or, $$i_1 = \frac{V_1(R_2+R_3+r_2)}{(R_1+R_3+r_1)(R_2+R_3+r_2)} + \frac{(-R_3V_2)}{(R_1+R_3+r_1)(R_2+R_3+r_2) - R_3^2}$$

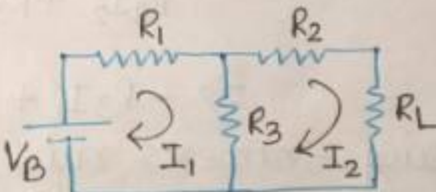
or, $\boxed{i_1 = i_1' + i_1''}$ (From equations ⑥ and ⑨)

Similarly, if we evaluate the values of i_2, i_2' and i_2'', then we must have, $\boxed{i_2 = i_2' + i_2''}$

Reciprocity theorem :-

It states that, in any linear network containing ohmic resistances and an energy source, the current in one loop when the emf acts in the other loop is the same as the current in the later when the same emf acts in the former.

Proof:-



Battery of voltage V_B is placed in the 1st loop.

Applying KVL in the 1st loop,

$$I_1R_1 + I_1R_3 - V_B - I_2R_3 = 0$$

$$\Rightarrow I_1(R_1+R_3) - I_2R_3 = V_B \quad \text{——— (1)}$$

and, applying KVL in the 2nd loop,

$$R_2I_2 + R_LI_2 + R_3I_2 - R_3I_1 = 0$$

$$\Rightarrow -R_3I_1 + (R_2+R_3+R_L)I_2 = 0 \quad \text{——— (2)}$$

Using Cramer's rule, the current in the 2nd loop.

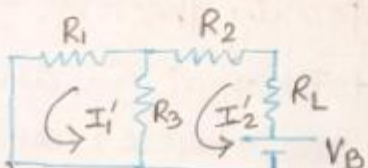
$$I_2 = \frac{\begin{vmatrix} (R_1+R_3) & V_B \\ -R_3 & 0 \end{vmatrix}}{\begin{vmatrix} (R_1+R_3) & -R_3 \\ -R_3 & (R_2+R_3+R_L) \end{vmatrix}}$$

$$\text{or, } I_2 = \frac{V_B R_3}{(R_1+R_3)(R_2+R_3+R_L)-R_3^2}$$

$$\Rightarrow I_2 = \frac{V_B R_3}{(R_1+R_3)(R_2+R_L)+R_1R_3} \quad \text{——— (3)}$$

Now, the battery is placed in the 2nd loop.

Applying KVL in 1st loop,



$$R_1I_1' + R_3I_1' - R_3I_2' = 0$$

$$\Rightarrow (R_1+R_3)I_1' - R_3I_2' = 0 \text{ — (4)}$$

and, applying KVL in the 2nd loop,

$$R_2I_2' + R_3I_2' - V_B + R_LI_2' - R_3I_1' = 0$$

$$\Rightarrow -R_3I_1' + (R_2+R_3+R_L)I_2' = V_B \quad \text{——— (5)}$$

Using Cramer's rule, the current in the 1st loop:

$$\therefore I_1' = \frac{\begin{vmatrix} 0 & -R_3 \\ V_B & (R_2+R_3+R_L) \end{vmatrix}}{\begin{vmatrix} (R_1+R_3) & -R_3 \\ -R_3 & (R_2+R_3+R_L) \end{vmatrix}}$$

$$\Rightarrow I_1' = \frac{R_3V_B}{(R_1+R_3)(R_2+R_3+R_L)-R_3^2} \quad \text{——— (6)}$$

From equations (3) & (6), we have, $I_2 = I_1'$, which establishes the reciprocity theorem

Maximum power transfer theorem:-

A load resistance may actually be connected directly across a battery with an internal resistance. It may also be connected to a circuit, as shown in fig-I, whence the circuit reduces to the thevenin's equivalent form of figure-II. Eventually then the load resistance R_L is connected to a battery of generated voltage V_g and internal resistance R_g. We shall vary R_L and determine the condition for which a maximum power is absorbed by the load resistance from the source.

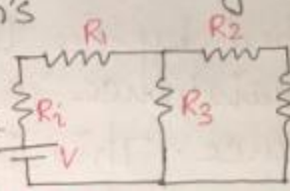


Fig: I A load resistance R_L connected to an ohmic circuit.

Fig-II Thevenin's equivalent Circuit

The load current is,

$$i = \frac{V_g}{(R_g + R_L)} \quad \text{——} (1)$$

So that, the power in the resistance R_L is,

$$P = i^2 R_L = \left[\frac{V_g}{R_g + R_L}\right]^2 R_L \quad \text{——} (2)$$

The variation of the dissipated power P as a functio of R_L is depicted in fig-III.

To find the value of R_L for which P is maximum, we put,

$$\frac{dP}{dR_L} = 0$$

or, $$\frac{(R_g+R_L)^2 - 2R_L(R_g+R_L)}{(R_g+R_L)^4} = 0$$

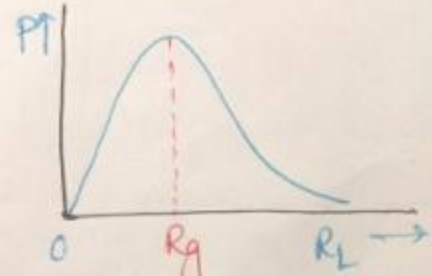


Fig-III Variation of P with R_L.

or, $R_g + R_L - 2R_L = 0$

or, $\boxed{R_L = R_g}$ ——— (3)

Thus, in a resistive circuit, if the load resistance is varied, maximum power is absorbed by the load resistance when the load resistance is equal to the source resistance. This is known as the maximum power transfer theorem.

Applications of dc circuits:-

There are widespread applications of dc circuits in the modern era of science and technology. Some of them are discussed in the following-

1. DC circuits are commonly found in low voltage applications, especially where these are powered by batteries or solar power systems used in domestic and commercial buildings.
2. Most electronic circuits require a DC power supply.
3. Most automotive applications use DC. An automotive battery provides power for engine starting, lighting, and ignition system.
4. Several dc circuits are used in telecommunication systems.

*** There may be some unintentional mistakes, if any, students are requested to address me those mistakes through the given email address.

*** To enhance the analytical handling capacity, students are advised to solve a lot of problems on "Network theorem", highly available on the examples and exercises of the reference books.