Chapter 2

First Order Ordinary Differential Equations

The complexity of solving DE's increases with the order. We begin with first order DE's.

2.1 Separable Equations

A first order ODE has the form $F(x, y, y') = 0$. In theory, at least, the methods of algebra can be used to write it in the form* $y' = G(x, y)$. If $G(x, y)$ can be factored to give $G(x, y) = M(x)\,N(y)$,then the equation is called *separable.* To solve the separable equation $y' = M(x)\,N(y)$, we rewrite it in the form $f(y)y' = g(x)$. Integrating both sides gives

$$\int f(y)y'\,dx = \int g(x)\,dx,$$
$$\int f(y)\,dy = \int f(y)\frac{dy}{dx}\,dx.$$

Example 2.1. Solve $2xy + 6x + \left(x^2 - 4\right)y' = 0$. $\ast$

*We use the notation $dy/dx = G(x, y)$ and $dy = G(x, y)\,dx$ interchangeably.

Solution. Rearranging, we have

$$\begin{aligned}
\left(x^2-4\right)y' &= -2xy-6x, \\
&= -2xy-6x, \\
\frac{y'}{y+3} &= -\frac{2x}{x^2-4}, \qquad x \neq \pm 2 \\
\ln(|y+3|) &= -\ln\left(\left|x^2-4\right|\right)+C, \\
\ln(|y+3|)+\ln\left(\left|x^2-4\right|\right) &= C,
\end{aligned}$$

where C is an arbitrary constant. Then

$$\begin{aligned}
\left|(y+3)\left(x^2-4\right)\right| &= A, \\
(y+3)\left(x^2-4\right) &= A, \\
y+3 &= \frac{A}{x^2-4},
\end{aligned}$$

where A is a constant (equal to $\pm e^C$) and $x \neq \pm 2$. Also $y=-3$ is a solution (corresponding to $A=0$) and the domain for that solution is $\mathbb{R}$. ◊

Example 2.2. Solve the IVP $\sin(x)\,dx + y\,dy = 0$, where $y(0)=1$. *

Solution. Note: $\sin(x)\,dx + y\,dy = 0$ is an alternate notation meaning the same as $\sin(x) + y\,dy/dx = 0$.

We have

$$\begin{aligned}
y\,dy &= -\sin(x)\,dx, \\
\int y\,dy &= \int -\sin(x)\,dx, \\
\frac{y^2}{2} &= \cos(x)+C_1, \\
y &= \sqrt{2\cos(x)+C_2},
\end{aligned}$$

where C_1 is an arbitrary constant and $C_2 = 2C_1$. Considering $y(0)=1$, we have

$$1 = \sqrt{2+C_2} \Longrightarrow 1 = 2 + C_2 \Longrightarrow C_2 = -1.$$

Therefore, $y = \sqrt{2\cos(x)-1}$ on the domain $(-\pi/3, \pi/3)$, since we need $\cos(x) \geq 1/2$ and $\cos(\pm\pi/3) = 1/2$.

An alternate method to solving the problem is

$$\begin{aligned}
y\,dy &= -\sin(x)\,dx, \\
\int_1^y y\,dy &= \int_0^x -\sin(x)\,dx, \\
\frac{y^2}{2} - \frac{1^2}{2} &= \cos(x) - \cos(0), \\
\frac{y^2}{2} - \frac{1}{2} &= \cos(x) - 1, \\
\frac{y^2}{2} &= \cos(x) - \frac{1}{2}, \\
y &= \sqrt{2\cos(x) - 1},
\end{aligned}$$

giving us the same result as with the first method. ♢

Example 2.3. Solve $y^4 y' + y' + x^2 + 1 = 0$. $*$

Solution. We have

$$\begin{aligned}
\left(y^4 + 1\right) y' &= -x^2 - 1, \\
\frac{y^5}{5} + y &= -\frac{x^3}{3} - x + C,
\end{aligned}$$

where C is an arbitrary constant. This is an implicit solution which we cannot easily solve explicitly for y in terms of x. ♢

2.2 Exact Differential Equations

Using algebra, any first order equation can be written in the form $F(x,y)\,dx + G(x,y)\,dy = 0$ for some functions $F(x,y)$, $G(x,y)$.

Definition

An expression of the form $F(x,y)\,dx + G(x,y)\,dy$ is called a *(first-order) differential form*. A differentical form $F(x,y)\,dx + G(x,y)\,dy$ is called *exact* if there exists a function $g(x,y)$ such that $dg = F\,dx + G\,dy$.

If $\omega = F\,dx + G\,dy$ is an exact differential form, then $\omega = 0$ is called an *exact differential equation*. Its solution is $g = C$, where $\omega = dg$.

Recall the following useful theorem from `MATB42`:

Theorem 2.4
If F and G are functions that are continuously differentiable throughout a simply connected region, then $F\,dx + G\,dy$ is exact if and only if $\partial G/\partial x = \partial F/\partial y$.

Proof. Proof is given in `MATB42`. □

Example 2.5. Consider $\left(3x^2y^2 + x^2\right)dx + \left(2x^3y + y^2\right)dy = 0$. Let

$$\omega = \underbrace{\left(3x^2y^2 + x^2\right)}_{F}dx + \underbrace{\left(2x^3y + y^2\right)}_{G}dy$$

Then note that

$$\frac{\partial G}{\partial x} = 6x^2y = \frac{\partial F}{\partial y}.$$

By Theorem 2.4, $\omega = dg$ for some g. To find g, we know that

$$\frac{\partial g}{\partial x} = 3x^2y^2 + x^2, \tag{2.1a}$$
$$\frac{\partial g}{\partial y} = 2x^3y + y^2. \tag{2.1b}$$

Integrating Equation (2.1a) with respect to x gives us

$$g = x^3y^2 + \frac{x^3}{3} + h(y). \tag{2.2}$$

So differentiating that with respect to y gives us

$$\begin{aligned}
\overbrace{\frac{\partial g}{\partial y}}^{\text{Eq. (2.1b)}} &= 2x^3y + \frac{dh}{dy}, \\
2x^3y + y^2 &= 2x^3y + \frac{dh}{dy}, \\
\frac{dh}{dy} &= y^2, \\
h(y) &= \frac{y^3}{3} + C
\end{aligned}$$

for some arbitrary constant C. Therefore, Equation (2.2) becomes

$$g = x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C.$$

Note that according to our differential equation, we have

$$d\left(x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C\right) = 0 \text{ which implies } x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C = C'$$

for some arbitrary constant C'. Letting $D = C' - C$, which is still an arbitrary constant, the solution is

$$x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = D.$$

*

Example 2.6. Solve $\left(3x^2 + 2xy^2\right) dx + \left(2x^2y\right) dy = 0$, where $y(2) = -3$. *

Solution. We have

$$\int \left(3x^2 + 2xy^2\right) dx = x^3 + x^2y^2 + C$$

for some arbitrary constant C. Since C is arbitrary, we equivalently have $x^3 + x^2y^2 = C$. With the initial condition in mind, we have

$$8 + 4 \cdot 9 = C \Longrightarrow C = 44.$$

Therefore, $x^3 + x^2y^2 = 44$ and it follows that

$$y = \frac{\pm\sqrt{44 - x^3}}{x^2}.$$

But with the restriction that $y(2) = -3$, the only solution is

$$y = -\frac{\sqrt{44 - x^3}}{x^2}$$

on the domain $\left(-\sqrt[3]{44}, \sqrt[3]{44}\right) \setminus \{0\}$. ♢

Let $\omega = F\,dx + G\,dy$. Let $y = s(x)$ be the solution of the DE $\omega = 0$, i.e., $F + Gs'(x) = 0$. Let $y_0 = s(x_0)$ and let γ be the piece of the graph of $y = s(x)$ from (x_0, y_0) to (x, y). Figure 2.1 shows this idea. Since $y = s(x)$ is a solution to $\omega = 0$, we must have $\omega = 0$ along γ. Therefore, $\int_\gamma \omega = 0$. This can be seen

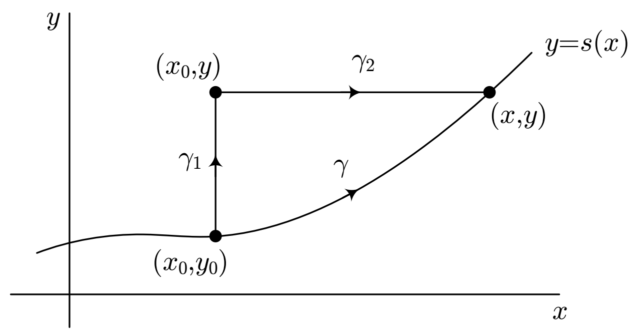

Figure 2.1: The graph of $y = s(x)$ with γ connecting (x_0, y_0) to (x, y).

by parameterizing γ by $\gamma(x) = (x, s(x))$, thereby giving us

$$\int_\gamma \omega = \int_{x_0}^{x} F\,dx + Gs'(x)\,dx = \int_{x_0}^{x} 0\,dx = 0.$$

This much holds for any ω.

Now suppose that ω is exact. Then the integral is independent of the path. Therefore

$$\begin{aligned} 0 = \int_\gamma \omega &= \int_{\gamma_1} F\,dx + G\,dy + \int_{\gamma_2} F\,dx + G\,dy \\ &= \int_{y_0}^{y} G(x_0, y)\,dy + \int_{x_0}^{x} F(x, y)\,dx. \end{aligned}$$

We can now solve Example 2.6 with this new method.

Solution (Alternate solution to Example 2.6). We simply have

$$\begin{aligned} 0 &= \int_{-3}^{4} 2 \cdot 2^2 y\,dy + \int_{2}^{x} \left(3x^2 + 2xy^2\right) dx \\ &= 4y^2 - 4\left(-3\right)^2 + x^3 + x^2y^2 - 2^3 - 2^2y^2 \\ &= 4y^2 - 36 + x^3 + x^2y^2 - 8 - 4y^2, \end{aligned}$$

finally giving us $x^3 + x^2y^2 = 44$, which agrees with our previous answer. $\Diamond$

Remark. Separable equations are actually a special case of exact equations, that is,

$$f(y)y' = g(x) \Longrightarrow -g(x)\,dx + f(y)\,dy = 0 \Longrightarrow \frac{\partial}{\partial x} f(y) = 0 = \frac{\partial}{\partial y}\left(-g(x)\right).$$

So the equation is exact. ♢

2.3 Integrating Factors

Consider the equation $\omega = 0$. Even if ω is not exact, there may be a function $I(x, y)$ such that $I\omega$ is exact. So $\omega = 0$ can be solved by multiplying both sides by I. The function I is called an *integrating factor* for the equation $\omega = 0$.

Example 2.7. Solve $y/x^2 + 1 + y'/x = 0$. $*$

Solution. We have

$$\left(\frac{y}{x^2} + 1\right) dx + \frac{1}{x}\, dy = 0.$$

We see that

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right] \neq \left[\frac{1}{x^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2} + 1\right)\right].$$

So the equation is not exact. Multiplying by x^2 gives us

$$\begin{aligned}
\left(y + x^2\right) dx + x\, dy &= 0, \\
d\left(xy + \frac{x^3}{3}\right) &= 0, \\
xy + \frac{x^3}{3} &= C
\end{aligned}$$

for some arbitrary constant C. Solving for y finally gives us

$$y = \frac{C}{x} - \frac{x^3}{3}.$$ ♢

There is, in general, no algorithm for finding integrating factors. But the following may suggest where to look. It is important to be able to recognize common exact forms:

$$\begin{aligned}
x\, dy + y\, dx &= d(xy)\,, \\
\frac{x\, dy - y\, dx}{x^2} &= d\left(\frac{y}{x}\right), \\
\frac{x\, dx + y\, dy}{x^2 + y^2} &= d\left(\frac{\ln\left(x^2 + y^2\right)}{2}\right), \\
\frac{x\, dy - d\, dx}{x^2 + y^2} &= d\left(\tan^{-1}\left(\frac{y}{x}\right)\right), \\
x^{a-1} y^{b-1} \left(ay\, dx + bx\, dy\right) &= d\left(x^a y^b\right).
\end{aligned}$$

Example 2.8. Solve $\left(x^2y^2+y\right)dx+\left(2x^3y-x\right)dy=0$. $\quad *$

Solution. Expanding, we have

$$x^2y^2\,dx+2x^3y\,dy+y\,dx-x\,dy=0.$$

Here, $a=1$ and $b=2$. Thus, we wish to use

$$d\left(xy^2\right)=y^2\,dx+2xy\,dy.$$

This suggests dividing the original equation by x^2 which gives

$$y^2\,dx+2xy\,dy+\frac{y\,dx-x\,dy}{x^2}=0.$$

Therefore,

$$xy^2+\frac{y}{x}=C,\quad x\neq 0,$$

where C is an arbitrary constant. Additionally, $y=0$ on the domain $\mathbb{R}$ is a solution to the original equation. $\quad \Diamond$

Example 2.9. Solve $y\,dx-x\,dy-\left(x^2+y^2\right)dx=0$. $\quad *$

Solution. We have

$$\frac{y\,dx-x\,dy}{x^2+y^2}-dx=0,$$

unless $x=0$ and $y=0$. Now, it follows that

$$\begin{aligned}
-\tan^{-1}\left(\frac{y}{x}\right)-x&=C,\\
\tan^{-1}\left(\frac{y}{x}\right)&=-C-x,\\
\tan^{-1}\left(\frac{y}{x}\right)&=D-x,\quad (D=-C)\\
\frac{y}{x}&=\tan(D-x)\,,\\
y&=x\tan(D-x)\,,
\end{aligned}$$

where C is an arbitrary constant and the domain is

$$D-x\neq(2n+1)\,\frac{\pi}{2},\quad x\neq(2n+1)\,\frac{\pi}{2}$$

for any integer n. Also, since the derivation of the solution is based on the assumption that $x\neq 0$, it is unclear whether or not 0 should be in the domain, i.e., does $y=x\tan(D-x)$ satisfy the equation when $x=0$? We have $y-xy'-$

$\left(x^2+y^2\right)=0$. If $x=0$ and $y=x\tan(D-x)$, then $y=0$ and the equation is satisfied. Thus, 0 is in the domain. ♢

Proposition 2.10
Let $\omega = dg$. Then for any function $P:\mathbb{R}\to\mathbb{R}$, $P(g)$ is exact.

Proof. Let $Q=\int P(t)\,dy$. Then $d(Q(g)) = P(g)\,dg = P(g)\omega$. □

To make use of Proposition 2.10, we can group together some terms of ω to get an expression you recognize as having an integrating factor and multiply the equation by that. The equation will now look like $dg+h=0$. If we can find an integrating factor for h, it will not necessarily help, since multiplying by it might mess up the part that is already exact. But if we can find one of the form $P(g)$, then it will work.

Example 2.11. Solve $\left(x-yx^2\right)dy + y\,dx = 0$. $*$

Solution. Expanding, we have

$$\underbrace{y\,dx + x\,dy}_{d(xy)} - yx^2\,dy = 0.$$

Therefore, we can multiply teh equation by any function of xy without disturbing the exactness of its first two terms. Making the last term into a function of y alone will make it exact. So we multiply by $(xy)^{-2}$, giving us

$$\frac{y\,dx + x\,dy}{x^2y^2} - \frac{1}{y}\,dy = 0 \Longrightarrow -\frac{1}{xy} - \ln(|y|) = C,$$

where C is an arbitrary constant. Note that $y=0$ on the domain $\mathbb{R}$ is also a solution. ♢

Given

$$M\,dx + N\,dy = 0, \tag{$*$}$$

we want to find I such that $IM\,dx + IN\,dy$ is exact. If so, then

$$\underbrace{\frac{\partial}{\partial x}(IN)}_{I_xN+IN_x} = \underbrace{\frac{\partial}{\partial y}(IM)}_{I_yM+IM_y}.$$

If we can find any particular solution $I(x,y)$ of the PDE

$$I_x N + I N_x = I_y M + I M_y, \tag{$**$}$$

then we can use it as an integrating factor to find the general solution of $(*)$. Unfortunately, $(**)$ is usually even harder to solve than $(*)$, but as we shall see, there are times when it is easier.

Example 2.12. We could look for an I having only x's and no y's? For example, consider $I_y = 0$. Then

$$I_x N + I N_x = I M_y \text{ implies } \frac{I_x}{I} = \frac{M_y - N_x}{N}.$$

This works if $(M_y - N_x)/N$ happens to be a function of x alone. Then

$$I = e^{\int \frac{M_y - N_x}{N}\,dx}.$$

Similarly, we can also reverse the role of x and y. If $(N_x - M_y)/M$ happens to be a function of y alone, then

$$e^{\int \frac{N_x - M_y}{M}\,dy}$$

works. $*$

2.4 Linear First Order Equations

A first order linear equation $(n = 1)$ looks like

$$y' + P(x)y = Q(x).$$

An integrating factor can always be found by the following method. Consider

$$dy + P(x)y\,dx = Q(x)\,dx,$$
$$\underbrace{(P(x)y - Q(x))}_{M(x,y)}\,dx + \underbrace{dy}_{N(x,y)} = 0.$$

We use the DE for the integrating factor $I(x,y)$. The equation $IM\,dx + IN\,dy$ is exact if

$$I_x N + I N_x = I_y M + I M_y.$$

In our case,

$$I_x + 0 = I_y\,(P(x)y - Q(x)) + IP(x). \tag{$*$}$$

We need only one solution, so we look for one of the form $I(x)$, i.e., with $I_y = 0$. Then $(*)$ becomes

$$\frac{dI}{dx} = IP(x).$$

This is separable. So

$$\begin{aligned}
\frac{dI}{I} &= P(x)\,dx, \\
\ln(|I|) &= \int P(x)\,dx + C, \\
|I| &= e^{\int P(x)\,dx}, \quad e^x > 0 \\
I &= e^{\int P(x)\,dx}.
\end{aligned}$$

We conclude that $e^{\int P(x)\,dx}$ is an integrating factor for $y' + P(x)y = Q(x)$.

Example 2.13. Solve $y' - (1/x)\,y = x^3$, where $x > 0$. $*$

Solution. Here $P(x) = -1/x$. Then

$$I = e^{\int P(x)\,dx} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln(|x|)dx} = \frac{1}{|x|} = \frac{1}{x},$$

where $x > 0$. Our differential equation is

$$\frac{x\,dy - y\,dx}{x} = x^3\,dx.$$

Multiplying by the integrating factor $1/x$ gives us

$$\frac{x\,dy - y\,dx}{x^2} = x^2\,dx.$$

Then

$$\begin{aligned}
\frac{y}{x} &= \frac{x^3}{3} + C, \\
y &= \frac{x^3}{3} + Cx
\end{aligned}$$

on the domain $(0, \infty)$, where C is an arbitrary constant ($x > 0$ is given). $\Diamond$

In general, given $y' + P(x)y = Q(x)$, multiply by $e^{\int P(x)\,dx}$ to obtain

$$\underbrace{e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x)y}_{d\left(ye^{\int P(x)\,dx}\right)/dx} = Q(x)e^{\int P(x)\,dx}.$$

Therefore,

$$\begin{aligned} ye^{\int P(x)\,dx} &= \int Q(x)e^{\int P(x)\,dx}\,dx + C, \\ y = e^{-\int P(x)\,dx} &\int Q(x)e^{\int P(x)\,dx}\,dx + Ce^{-\int P(x)\,dx}, \end{aligned}$$

where C is an arbitrary constant.

Example 2.14. Solve $xy' + 2y = 4x^2$. $*$

Solution. What should $P(x)$ be? To find it, we put the equation in standard form, giving us

$$y' + \frac{2}{x}y = 4x.$$

Therefore, $P(x) = 2/x$. Immediately, we have

$$I = e^{\int (2/x)dx} = e^{\ln\left(x^2\right)} = x^2.$$

Multiplying the equation by x^2 gives us

$$\begin{aligned} x^2y' + 2xy &= 4x^3, \\ x^2y &= x^4 + C, \\ y &= x^2 + \frac{C}{x^2}, \end{aligned}$$

where C is an arbitrary constant and $x \neq 0$. $\Diamond$

Example 2.15. Solve $e^{-y}\,dy + dx + 2x\,dy = 0$. $*$

Solution. This equation is linear with x as a function of y. So what we have is

$$\frac{dx}{dy} + 2x = -e^{-y},$$

where $I = e^{\int 2\,dy} = e^{2y}$. Therefore,

$$e^{2y}\frac{dx}{dy} + 2xe^{2y} = -e^{y},$$
$$xe^{2y} = -e^{y} + C,$$

where C is an arbitrary constant. We could solve explicitly for y, but it is messy. The domain is not easy to determine. ♢

2.5 Substitutions

In many cases, equations can be put into one of the standard forms discussed above (separable, linear, etc.) by a substitution.

Example 2.16. Solve $y'' - 2y' = 5$. *

Solution. This is a first order linear equation for y'. Let $u = y'$. Then the equation becomes

$$u' - 2u = 5.$$

The integration factor is then $I = e^{-\int 2\,dx} = e^{-2x}$. Thus,

$$u'e^{-2x} - 2ue^{-2x} = 5e^{-2x},$$
$$ue^{-2x} = -\frac{5}{2}e^{-2x} + C,$$

where C is an arbitrary constant. But $u = y'$, so

$$y = -\frac{5}{2}x + \frac{C}{2}e^{2x} + C_1 = -\frac{5}{2}x + C_1e^{2x} + C_2$$

on the domain $\mathbb{R}$, where C_1 and C_2 are arbitrary constants. ♢

We now look at standard substitutions.

2.5.1 Bernoulli Equation

The Bernoulli equation is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^{n}.$$

Let $z = y^{1-n}$. Then

$$\frac{dz}{dx} = (1-n)\,y^{-n}\frac{dy}{dx},$$

giving us

$$\begin{aligned}
y^{-n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x), \\
\frac{1}{1-n}\frac{dz}{dx} + P(x)z &= Q(x), \\
\frac{dz}{dx} + (1-n)\,P(x)z &= (1-n)\,Q(x),
\end{aligned}$$

which is linear in z.

Example 2.17. Solve $y' + xy = xy^3$. $\ast$

Solution. Here, we have $n = 3$. Let $z = y^{-2}$. If $y \neq 0$, then

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}.$$

Therefore, our equation becomes

$$\begin{aligned}
-\frac{y^3 z'}{2} + xy &= xy^3, \\
-\frac{z'}{2} + xy^{-2} &= x, \\
z' - 2xy &= -2x.
\end{aligned}$$

We can readily see that $I = e^{-\int 2x\,dx} = e^{-x^2}$. Thus,

$$\begin{aligned}
e^{-x^2}z' - 2xe^{-x^2} &= -2xe^{-x^2}, \\
e^{-x^2}z &= e^{-x^2} + C, \\
z &= 1 + Ce^{x^2},
\end{aligned}$$

where C is an arbitrary constant. But $z = y^{-2}$. So

$$y = \pm\frac{1}{\sqrt{1 + Ce^{x^2}}}.$$

The domain is

$$\begin{cases} \mathbb{R}, & C > -1, \\ |x| > \sqrt{-\ln(-C)}, & C \leq -1. \end{cases}$$

An additional solution is $y = 0$ on $\mathbb{R}$. $\lozenge$

2.5.2 Homogeneous Equations

Definition (Homogeneous function of degree n)
A function $F(x, y)$ is called *homogeneous of degree* n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. For a polynomial, homogeneous says that all of the terms have the same degree.

Example 2.18. The following are homogeneous functions of various degrees:

$$\begin{aligned}
3x^6 + 5x^4y^2 &\qquad \text{homogeneous of degree 6,} \\
3x^6 + 5x^3y^2 &\qquad \text{not homogeneous,} \\
x\sqrt{x^2+y^2} &\qquad \text{homogeneous of degree 2,} \\
\sin\left(\frac{y}{x}\right) &\qquad \text{homogeneous of degree 0,} \\
\frac{1}{x+y} &\qquad \text{homogeneous of degree } -1.
\end{aligned}$$

*

If F is homogeneous of degree n and G is homogeneous of degree k, then F/G is homogeneous of degree $n-k$.

Proposition 2.19
If F is homogeneous of degree 0, then F is a function of y/x.

Proof. We have $F(\lambda x, \lambda y) = F(x, y)$ for all λ. Let $\lambda = 1/x$. Then $F(x, y) = F(1, y/x)$. □

Example 2.20. Here are some examples of writing a homogeneous function of degree 0 as a function of y/x.

$$\begin{aligned}
\frac{\sqrt{5x^2+y^2}}{x} &= \sqrt{5 + \left(\frac{y}{x}\right)^2}, \\
\frac{y^3 + x^2y}{x^2y + x^3} &= \frac{(y/x)^3 + (y/x)}{(y/x) + 1}.
\end{aligned}$$

*

Consider $M(x, y)\,dx + N(x, y)\,dy = 0$. Suppose M and N are both homogeneous and of the same degree. Then

$$\frac{dy}{dx} = -\frac{M}{N},$$

This suggests that $v = y/x$ (or equivalently, $y = vx$) might help. In fact, write

$$-\frac{M(x,y)}{N(x,y)} = R\Big(\frac{y}{x}\Big).$$

Then

$$\underbrace{\frac{dy}{dx}}_{v+x\frac{dv}{dx}} = R\Big(\frac{y}{x}\Big) = R(v).$$

Therefore,

$$\begin{aligned} x\frac{dv}{dx} &= R(v) - v, \\ \frac{dv}{R(v)-v} &= \frac{dx}{x}, \end{aligned}$$

which is separable. We conclude that if M and N are homogeneous of the same degree, setting $y = vx$ will give a separable equation in v and x.

Example 2.21. Solve $xy^2\,dy = \left(x^3 + y^3\right) dx$. $*$

Solution. Let $y = vx$. Then $dy = v\,dx + x\,dv$, and our equation becomes

$$\begin{aligned} xv^2x^2\left(v\,dx + x\,dv\right) &= \left(x^3 + v^3x^2\right) dx, \\ x^3v^3\,dx + x^4v^2\,dv &= x^3\,dx + v^3x^3\,dx. \end{aligned}$$

Therefore, $x = 0$ or $v^2\,dv = dx/x$. So we have

$$\frac{v^3}{3} = \ln(|x|) + C = \ln(|x|) + \underbrace{\ln(|A|)}_{C} = \ln(|Ax|) = \ln(Ax)\,.$$

where the sign of A is the opposite of the sign of x. Therefore, the general solution is $y = x\left(3\ln(Ax)\right)^{1/3}$, where A is a nonzero constant. Every $A > 0$ yields a solution on the domain $(0,\infty)$; every $A < 0$ yields a solution on $(-\infty, 0)$. In addition, there is the solution $y = 0$ on the domain $\mathbb{R}$. $\diamondsuit$

2.5.3 Substitution to Reduce Second Order Equations to First Order

A second order DE has the form

$$F(y'', y', y, x) = 0.$$

If it is independent of y, namely, $F(y'', y', x) = 0$, then it is really just a first order equation for y' as we saw in earlier examples.

Consider now the case where it is independent of x, namely, $F(y'', y', y) = 0$. Substitute $v = dy/dx$ for x, i.e., eliminate x between the equations

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0$$

and $v = dy/dx$. Then

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v.$$

Therefore,

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0 \rightsquigarrow F\left(\frac{dv}{dy}v, v, y\right) = 0.$$

This is a first order equation in v and y.

Example 2.22. Solve $y'' = 4\,(y')^{3/2}\,y$. *

Solution. Let $v = dy/dx$. Then

$$\frac{d^2y}{dx^2} = \frac{dv}{dy}v$$

and our equation becomes

$$\begin{aligned}
\frac{dv}{dy}v &= 4v^{3/2}y, \\
\frac{dv}{\sqrt{v}} &= 4y\,dy, \quad v \geq 0, \\
2\sqrt{v} &= 2y^2 + C_1, \\
\sqrt{v} &= y^2 + C_2,
\end{aligned}$$

where C_1 is an arbitrary constant and $C_2 = C_1/2$. But $v = dy/dx$, so we have

$$\begin{aligned}
\frac{dy}{dx} &= \left(y^2 + C_2\right)^2, \\
dx &= \frac{dy}{\left(y^2 + C_2\right)^2}, \\
x &= \int \frac{dy}{\left(y^2 + C_2\right)^2} \\
&= \begin{cases} \frac{1}{2C_2^{3/2}} \left(\tan^{-1}\left(\frac{y}{\sqrt{C_2}}\right) + \frac{\sqrt{C_2}y}{y^2+C_2}\right) + C_3, & C_2 > 0, \\ -\frac{1}{3y^3} + C_3, & C_2 = 0, \\ -\frac{1}{2(-C_2)^{3/2}} \cdot \frac{y^2}{y^2+C_2} + C_3, & C_2 < 0. \end{cases}
\end{aligned} \qquad \Diamond$$

Next consider second order linear equations. That is,

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

We can eliminate y by letting $y = e^v$. Then $y' = e^v v'$ and $y'' = e^v \left(v'\right)^2 + e^v v''$. The equation then becomes

$$P(x)\left(e^v \left(v'\right)^2 + e^v v''\right) + Q(x)e^v v' + R(x)e^v = 0,$$

which is a first order equation in v'.

Example 2.23. Solve $x^2y'' + \left(x - x^2\right)y' - e^{2x}y = 0$. $\qquad *$

Solution. Let $y = e^v$. Then the equation becomes

$$x^2e^v \left(v'\right)^2 + x^2e^v v'' + \left(x - x^2\right)e^v v' - e^{2x}e^v = 0.$$

Write $z = v'$. Then

$$x^2z' + x^2z^2 + (1 - x)\,xz = e^{2x}.$$

Now we are on our own—there is no standard technique for this case. Suppose we try $u = xy$. Then $z = u/x$ and

$$z' = -\frac{u}{x^2} + \frac{1}{x}u'.$$

Then it follows that

$$xu' + u^2 - xu = e^{2x}.$$

This is a bit simpler, but it is still nonstandard. We can see that letting $u = se^x$

will give us some cancellation. Thus, $u' = s'e^x + se^x$ and our equation becomes

$$\begin{aligned}
xs'e^x + \cancel{xse^x} + s^2e^{2x} - \cancel{xse^x} &= e^{2x}, \\
xs' + s^2e^x &= e^x, \\
xs' &= e^x\left(1 - s^2\right), \\
\frac{s'}{1-s^2} &= \frac{e^x}{x}, \\
\frac{1}{2}\ln\left(\left|\frac{1+s}{1-s}\right|\right) &= \int \frac{e^x}{x}\,dx.
\end{aligned}$$

Working our way back through the subsitutions we find that $s = zxe^{-x}$ so our solution becomes

$$\frac{1}{2}\ln\left(\left|\frac{1+zxe^{-x}}{1-zxe^{-x}}\right|\right) = \int \frac{e^x}{x}\,dx.$$

Using algebra, we could solve this equation for z in terms of x and then integrate the result to get v which then determines $y = e^v$ as a function of x. The algebraic solution for z is messy and we will not be able to find a closed form expression for the antidervative v, so we will abandon the calculation at this point. In practical applications one would generally use power series techniques on equations of this form and calculate as many terms of the Taylor series of the solution as are needed to give the desired accuracy. $\diamondsuit$

Next consider equations of the form

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0.$$

If $c_1 = c_2 = 0$, the equation is homogeneous of degree 1. If not, try letting $\bar{x} = x - h$ and $\bar{y} = y - k$. We try to choose h and k to make the equation homogeneous. Since h and k are constants, we have $d\bar{x} = dx$ and $d\bar{y} = dy$. Then our equation becomes

$$(a_1\bar{x} + a_1h + b_1\bar{y} + b_1k + c_1)\,d\bar{x} + (a_2\bar{x} + a_2h + b_2\bar{y} + b_2k + c_2)\,d\bar{y} = 0.$$

We want $a_1h + b_1k = -c_1$ and $a_2h + b_2k = -c_2$. We can always solve for h and k, unless

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

So suppose

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Then $(a_2, b_2) = m\,(a_1, b_1)$. Let $z = a_1 x + b_1 y$. Then $dz = a_1\,dx + b_1\,dy$. If $b_1 \neq 0$, we have

$$\begin{aligned} dy &= \frac{dz - a_1\,dx}{b_1}, \\ (z + c_1)\,dx + (mz + c_2)\,\frac{dz - a_1\,dx}{b_1} &= 0, \\ \left(z + c_1 + \frac{a_1}{b_1}\right) dx + \left(\frac{mz + c_2}{b_1}\right) dx &= 0, \\ b_1\,dx &= -\frac{mz + c_2}{z + c_1 + a_1/b_1}\,dz. \end{aligned}$$

This is a separable equation.

If $b_1 = 0$ but $b_2 \neq 0$, we use $z = a_2 x + b_2 y$ instead. Finally, if both $b_1 = 0$ and $b_2 = 0$, then the original equation is separable.