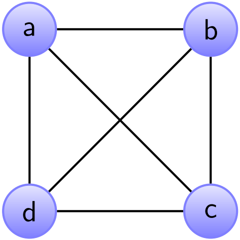
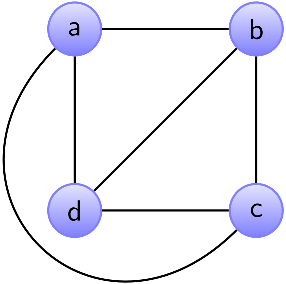
Planar Graphs

Definition

A simple connected graph is **planar** if it can be drawn in the plane without any edge crossings. Such a drawing is called an **embedding** of the graph in the plane.

The graph K_4, shown on the left, is planar because it can be drawn in the plane without any edge crossings.





Bipartite Graphs

Recall that:

Definition

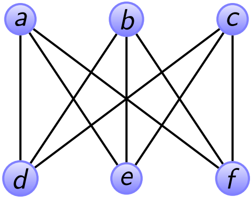
A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets S_1 and S_2 so that every edge in the graph has one endpoint in S_1 and one endpoint in S_2.

Definition

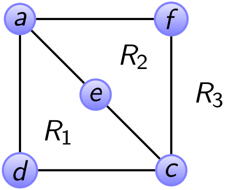
The **complete bipartite graph** on n, m nodes, denoted $K_{n,m}$, is the simple bipartite graph with nodes $S_1 = \{a_1, \ldots, a_n\}$ and $S_2 = \{b_1, \ldots, b_m\}$ and with edges connecting each node in S_1 to every node in S_2.

Is $K_{3,3}$ Planar?

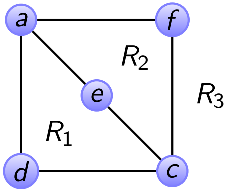
Is $K_{3,3}$ planar?



Suppose we start with a subgraph of $K_{3,3}$, which is planar. The following subgraph will partition the plane into three regions, R_1, R_2, and R_3.



Is $K_{3,3}$ Planar?



- If node b is put in R_1 it can be connected to d and e with no edge crossings but there is no way to connect it to f without a crossing.
- Similarly, if b is placed in R_2 we must cross an existing edge to get to d.
- Finally, if b is placed in R_3, we must cross an edge to get to e.

Similar investigations with other subgraphs will show that there is no way to draw $K_{3,3}$ without edge crossings, thus $K_{3,3}$ **is nonplanar**.

Euler's Formula

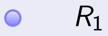
Euler showed that all planer representations of a graph partition the plane into the same number of regions.

Theorem (Euler's Formula)

If $G = (V, E)$ is a connected planar graph with r regions, v vertices, and e edges, then $v + r - e = 2$.

Proof.

Our proof is by induction. Starting with subgraph of G consisting a single vertex, we'll add edges and vertices (keeping the subgraphs connected) until we construct G. The basis step is



Here $r = 1$, $v = 1$, $e = 0$ so $r + v = e - 2$ becomes $1 + 1 = 0 + 2$, which is true.

(Continued next slide)

Euler's Formula

Proof.

(Continued) For the inductive step, assume that we have a subgraph of G with $e = k$ edges and that $r + v = e + 2$ holds. We now draw the the $k + 1^{st}$ edge. There are two cases:

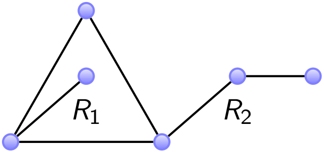
1. Both vertices the new edge is incident to are already on the graph. Since the subgraph is connected, this will create a new region. Thus both r and e increase by one so $r + v = e + 2$ is still true.
2. A new pendant vertex is introduced with the new edge. This does not increase r but does increase both e and v by one, so $r + v = e + 2$ continues to hold.

Since we can continue with these steps until we have constructed G, we conclude that $r + v = e + 2$ holds for any connected planar graph. □

Degrees of Regions

Definition

The **degree** of a region of a planar graph is the number of edge traversals required to trace the region's boundary.



- $\deg(R_1) = 5$
- $\deg(R_2) = 7$.

Question: What is the relationship between the number of edges in a graph and the sum of the degrees of the regions formed by the graph?

Answer: Each edge will be traversed twice, so

$$2e = \sum_{k=1}^{r} \deg(R_k).$$

A Simple Test for Nonplanarity

Question: Suppose G is a connected planar graph with at least three vertices. What is the minimum value of $\deg(R)$ for any region created by G?

Answer: $\deg(R_k) \geq 3$ for $k = 1, \ldots, r$.

In this case

$$2e = \sum_{k=1}^{r} \deg(R_k) \geq 3r = 3(e - v + 2)$$

so

$$\begin{aligned} 2e &\geq 3e - 3v + 6 \\ -e &\geq -3v + 6 \\ e &\leq 3v - 6. \end{aligned}$$

A Simple Test for Nonplanarity

This last result gives a useful test for nonplanarity.

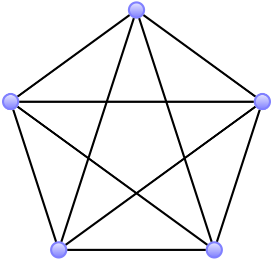
Theorem

Given a a connected planar graph G with $v \geq 3$ vertices and e edges, $e \leq 3v - 6$.

Thus, if $e > 3v - 6$ for a connected graph G with $v \geq 3$, we know that G is nonplanar.

Is K_5 Planar?

Is K_5 planar?



- Clearly $v = 5$ so $3v - 6 = 15 - 6 = 9$.
- We recall $e = C(5,2) = 10$ so $e > 3v - 6$ and we can conclude that K_5 is nonplanar.

Kuratowski's Theorem

Definition

An **elementary subdivision** in a graph G replaces any edge $\{u, v\}$ with a new vertex w and two new edges $\{u, w\}$ and $\{w, v\}$.

Definition

Two graphs are **homeomorphic** if they can be obtained from the same graph via elementary subdivisions.

Theorem (Kuratowski's Theorem)

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ *or* K_5.