# Theorems

## Theorem

*Let $G$ be a connected graph. Then $G$ is Eulerian if and only if every vertex in $G$ has even degree.*

## Theorem (Handshaking Lemma)

*In any graph with $n$ vertices $v_i$ and $m$ edges*

$$\sum_{i=1}^{n} \deg(v_i) = 2m$$

## Corollary

*A connected non-Eulerian graph has an Eulerian trail if and only if it has exactly two vertices of odd degree. The trail begins and ends these two vertices.*

# Theorems

## Theorem

*If $T$ is a tree with $n$ edges, then $T$ has $n+1$ vertices.*

## Theorem

*Two graphs that are isomorphic to one another must have*

1. *The same number of nodes.*
2. *The same number of edges.*
3. *The same number of nodes of any given degree.*
4. *The same number of cycles.*
5. *The same number of cycles of any given size.*

# Theorems

## Theorem (Kuratowski's Theorem)

*A graph $G$ is nonplanar if and only if it contains a "copy" of $K_{3,3}$ or $K_5$ as a subgraph.*

## Theorem (Euler's Formula for Planar Graphs)

*For any connected planar graph $G$ embedded in the plane with $V$ vertices, $E$ edges, and $F$ faces, it must be the case that*

$$V + F = E + 2.$$

# Graphs vs Plots

Recall that a graph consists of two sets: a set of vertices and a set of edges.

While we often represent graphs visually, we can distinguish between a graph and a **plot** in the following way: A graph stores information and connections between information while a plot provides a visual representation of the information stored in a graph.

Given that graphs are important, we now examine how we can represent graphs using a computer and see how one computer package handles graphs.

# A Quick Matrix Review

A **matrix** is a rectangular array of numbers. A matrix with $m$ **rows** and $n$ **columns** said to be an $m \times n$ matrix.

Entries in the matrix are addressed by their row and column numbers. Given a matrix $A$, the entry $a_{ij}$ is in the $i^{th}$ row and $j^{th}$ column of $A$. Notice that we always list the row index first.

We say a matrix $A$ is **symmetric** if $a_{ji} = a_{ij}$.

Not Symmetric

$$\begin{bmatrix} 0 & 5 & 2 & 1 \\ 1 & 3 & 0 & 1 \\ 4 & 6 & 8 & 3 \\ 0 & 7 & 3 & 1 \end{bmatrix}$$

Symmetric

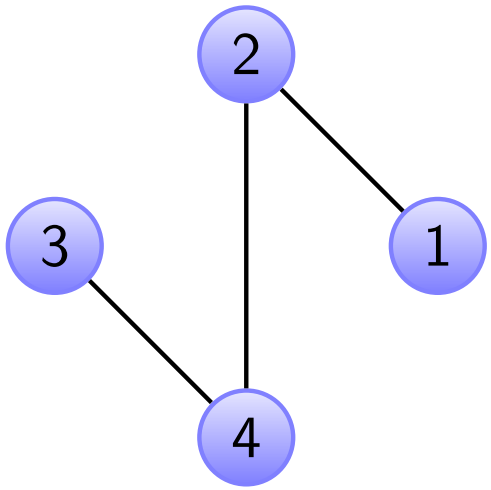
$$\begin{bmatrix} 0 & 3 & 7 & 1 \\ 3 & 8 & 5 & 0 \\ 7 & 5 & 2 & 4 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

# Adjacency Matrices

Let $G$ be a graph with $n$ vertices. We can use an $n \times n$ matrix to store the graph. Let

$$g_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{if vertex } i \text{ is not adjacent to vertex } j \end{cases}$$

For example, the graph on the left has the adjacency matrix on the right.




$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Note: matrix is symmetric

The adjacency matrix for a directed graph will not be symmetric unless the directed graph itself is symmetric.

# Sparse Graphs and Matrices

Consider $K_{30}$, the complete graph with 30 vertices. This graph has $C(30, 2) = 435$ edges since every vertex is connected to every other vertex. The adjacency matrix will have 1's in every non-diagonal position (why not on the diagonals?). We say the $30 \times 30$ adjacency matrix is **dense** or **full** since most of the entries are non-zero.

Now consider $C_{30}$, a **cycle** (or **ring**) graph with 30 vertices. Each vertex is connected to two other vertices to form a single ring or cycle. This means there are only 30 edges. So, while the adjacency matrix will be $30 \times 30$, only 60 entries in it will be non-zero. In this case we say the graph and the adjacency matrix are **sparse**.

# Adjacency Matrix Examples

Adjacency matrix for $K_8$

$8 \times 8$ matrix with 64 elements

$2 \cdot C(8,2) = 56$ non-zero entries

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency matrix for $C_8$

$8 \times 8$ matrix with 64 elements

8 non-zero entries

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$