Raniganj Girls' College

Department of Mathematics

5^{th} semester(Honours)

Subject-Mathematics

Pape-CC 11

Some Important Question

1. questions of 2 marks each

(a) If (X,d) is a connected metric space, prove that either X is a single-pointic set or it is an infinite set.

(b) If d is a metric on X, then show that $\dfrac{d}{1+d}$ is also a metric on X.

(c) If $\{x_n\}$ and $\{y_n\}$ are Cachy sequences in a metric space (X,d) then show that $\{d(x_n, y_n)\}$ is a convergent sequence.

(d) Let $f : c[a,b] \to R$ (the space of reals with usual metric) be defined by $f(x) = x(t_0)$, where t_0 is a fixed real number. Prove that f is continuous.

(e) Give an example to show that intersection if an infinite number of non-empty open sets is not an open set.

(f) For any complex number z, show that $\left|e^z\right| \le e^{|z|}$.

(g) Find all the limiting points of the sequence $\{z_n\}$ where $z_n = (-1)^n + \dfrac{n}{n+1} i$.

2. questions of 5 marks each

a. Let (X,d) be a metric space, and let

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$$, for all x, y in X.

Then prove that d^* is a bounded metric on X which is equivalent to d.

(b) Defined Fixed-point mapping. Let T be a mapping from a metric space (X,d) to itself. Prove that If T is contraction on X, then T is continuous on X

(c) Prove that in a metric space (X,d), a subset $F \subset M$ is closed if and only if its complement is open.

(d)Define closed set. If $|a| < 1$, show that $\displaystyle\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a$.

(e) If $f(z)=u(x,y)+iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, show that $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ at (x_0, y_0).

(f) (i) Show that an analytic function with constant modulus in a domain is constant.

(ii) Examine the differentiability of the function

$$f(z)=\begin{cases} xy^2(x+iy), \; if \; z\neq 0 \\ 0 \quad , \qquad if \; z=0 \end{cases}$$

at $z=0$. 1+4

(g) (i) Prove that every function continuous over $[a,b]$ is Riemann integrable. Is the converse true? Answer with reasons.

(ii) Let

$$f(x)=\begin{cases} x, if \; x \, is \, a \, rational \, number \, in [0,1] \\ 0, if \; x \, is \, an \, irrational \, number \, in [0,1] \end{cases}$$

Examine whether $\int_0^1 f(x)dx$ exists in Riemann sense.

(h) Let X be an infinite set with the discrete metric. Show that (X,d) is not compact

(i) Define pointwise convergence and uniform convergence of a sequence of functions. Prove that uniform convergence implies pointwise convergence but not conversely.

(j) Let X be the set of all continuous real-valued functions defined on $[0,1]$,

and let $d(x,y)=\int_0^1 |x(t)-y(t)|dt, \; \forall \; x,y\in X$. Show that (X,d) is not complete.

(k) Define closed set. Prove that in a metric space (X,d), a subset $F\subset M$ is closed if and only if its complement is open.